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Théorie du pluripotentiel et géométrie complexe

Habilitation à diriger des recherches de l'Université Paris-Saclay

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Habilitation à diriger des recherches

Pluripotential theory and complex geometry

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(2) V. Guedj, C.H. Lu. Quasi-plurisubharmonic envelopes 2: Bounds on Monge-Ampère volumes. arXiv:2106.04272.
(3) V. Guedj, C.H. Lu. Quasi-plurisubharmonic envelopes 1: Uniform estimates on Kähler manifolds. arXiv:2106.04273.
(4) E. Di Nezza, V. Guedj, C.H. Lu. Finite entropy vs finite energy. Comment. Math. Helv. 96 (2021), no. 2, 389–419.
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(6) C.H. Lu, T.T. Phung, T.D. Tô. Stability of complex Monge-Ampère equations on compact complex manifolds. arXiv:2003.08417, To appear in Ann. Inst. Fourier.
(7) T. Darvas, E. Di Nezza, C.H. Lu. The metric geometry of singularity types. J. Reine Angew. Math. 771 (2021), 137–170.
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(14) T. Darvas, C. H. Lu, Y. A. Rubinstein. Quantization in geometric pluripotential theory. Commun. Pure Appl. Math. 73, No. 5, 1100–1138 (2020).
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(22) T. Darvas, E. Di Nezza, C.H. Lu. Monotonicity of non-pluripolar products and complex Monge-Ampère equations with prescribed singularity. Anal. PDE 11 (2018), 2049–2087.
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Autres travaux

Les articles suivants ne sont pas présentés dans ce mémoire; le thème (équations Hessiennes) fait suite à ma thèse soutenue en 2012 sous la direction d'Ahmed Zeriahi:

- C.H. Lu, V.D. Nguyen. Complex Hessian equations with prescribed singularity on compact Kähler manifolds. arXiv:1909.02469. To appear in Ann. Sc. Norm. Super. Pisa.
- C.H. Lu, V.D. Nguyen, Degenerate complex Hessian equations on compact Kähler manifolds. Indiana Univ. Math. J. 64 (2015), no. 6, 1721–1745.
- S. Dinew, C.H. Lu. Mixed Hessian inequalities and uniqueness in the class $\mathcal{E}(X,\omega,m)$. Math. Z. 279 (2015), no. 3-4, 753–766.
- C.H. Lu. A variational approach to complex Hessian equations in $\mathbb{C}^n$. J. Math. Anal. Appl. 431 (2015), no. 1, 228–259.
- C.H. Lu. Solutions to degenerate complex Hessian equations. J. Math. Pures Appl. (9) 100 (2013), no. 6, 785–805.
- C.H. Lu. Viscosity solutions to complex Hessian equations. J. Funct. Anal. 264 (2013), no. 6, 1355–1379.

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Introduction

Ce mémoire présente un ensemble de résultats en analyse et géométrie complexe que j'ai obtenus après ma thèse en 2012, en collaboration avec plusieurs auteurs. Dans ces résultats on voit le rôle important de la théorie du pluripotentiel dans les développements récents en géométrie complexe. Le mémoire se décompose en quatre parties que l'on va maintenant décrire un peu en détail.

Équations de Monge-Ampère complexes à singularité prescrite

Soit X une variété kählérienne compacte de dimension n. Calabi a conjecturé en 1954 que dans chaque classe de cohomologie d'une métrique kählérienne ω on peut trouver une métrique à courbure de Ricci prescrite. Le problème se ramène à résoudre une équation de Monge-Ampère complexe, une EDP non-linéaire elliptique du second ordre. La résolution de cette conjecture par Aubin [**Aub78**] et Yau [**Yau78**] est basée sur la méthode de continuité et les estimées à priori, la partie la plus délicate étant l'estimée à priori L^∞ établie par Yau via le processus d'itération de Moser.

Dans le même temps, Bedford et Taylor [**BT76, BT82**] ont construit les premiers éléments d'une théorie des solutions faibles pour l'équation de Monge-Ampère complexe. Basé sur cette théorie, Kołodziej [**Koł98**] a mis au point une approche pluripotentielle pour établir l'estimée L^∞ valable dans plusieurs contextes géométriques. Tandis que la définition de l'opérateur de Monge-Ampère de Bedford et Taylor s'applique à toutes fonctions plurisousharmoniques bornées, étendre cette définition pour les fonctions non-bornées est un travail délicat. Cegrell [**Ceg98, Ceg04**] a introduit plusieurs classes contenant des fonctions psh non-bornées pour lesquelles on peut définir convenablement l'opérateur de Monge-Ampère. Contrairement au cas local considéré par Cegrell, sur une variété compacte kählérienne la masse de Monge-Ampère est facile à contrôler. Guidés par cette observation, Guedj-Zeriahi [**GZ07**], et Boucksom-Eyssidieux-Guedj-Zeriahi [**BEGZ10**] ont introduit la notion de "produit non-pluripolaire". Pendant plus de vingt ans la théorie s'est développée intensivement et elle a connu beaucoup d'applications géométriques remarquables.

Le premier chapitre de ce mémoire se place dans ce contexte. Il regroupe notamment des travaux obtenus en collaboration avec T. Darvas et E. Di Nezza [**DNL17a**], [**DDNL18c, DDNL18b, DDNL21a**] dont le thème principal est la résolution de l'équation de Monge-Ampère complexe dans un contexte d'une classe de cohomologie grosse. Dans les Théorème 1.10 et Théorème 1.11 on établit des estimées à priori L^∞ relatives généralisant l'estimée de Yau [**Yau78**] et Kołodziej [**Koł98**]. L'outil principal dans notre approche est la capacité de Monge-Ampère généralisée. Le thème central dans les travaux [**DDNL18c, DDNL18b, DDNL21a**] est la théorie du pluripotentiel relative. En particulier, on résout l'équation de Monge-Ampère complexe à singularité prescrite par un potentiel modèle (Théorème 1.12).

Un élément indispensable dans nos travaux est la monotonie de la masse du produit non-pluripolaire par rapport au type de singularité du potentiel (Théorème 1.1). En utilisant la résolution de cette équation, on montre que la fonction volume, définie sur le cône des courants fermés positifs de type $(1, 1)$, est log-concave (Théorème 1.13), confirmant une conjecture de Boucksom-Eyssidieux-Guedj-Zeriahi [**BEGZ10**].

Théorie du pluripotentiel sur des variétés hermitiennes compactes

Le deuxième chapitre concerne des travaux en collaboration avec T.T. Phung et T.D. Tô [**LPT20**], et avec V. Guedj [**GL21a, GL21b, GL21c**]. Nous nous somme intéressés aux équations de Monge-Ampère complexes sur des variétées hermitiennes compactes (non-kählériennes). Le sujet a été initié par Cherrier [**Che87**] et Hanani [**Han96**], qui ont essayé de démontrer une version analogue du théorème de Yau [**Yau78**]. La stratégie est la même que dans le cas kählérien: on utilise la méthode de continuité et on établit les estimées a priori. Dans [**Che87**] l'existence d'une solution lisse a été démontrée sous une condition assez restrictive sur la forme de référence, utilisée pour établir une estimée L^∞. Cette dernière a été enfin obtenue en toute généralité par Tosatti-Weinkove [**TW10**]. La difficulté du problème vient du fait que la forme de référence n'est pas fermée, ce qui produit plusieurs termes à contrôler à chaque fois que l'on utilise le théorème de Stokes.

L'intérêt pour la théorie pluripotentielle dans ce cadre a grandi pendant la dernière décennie. S. Dinew, S. Kołodziej, N.C. Nguyen ont posé les premiers pierres de cette théorie et ont obtenu quelques résultats importants. Dans [**LPT20**] (Théorème 2.7 et Théorème 2.8) nous avons amélioré les résultats de stabilité et régularité de Kołodzeij-Nguyen [**KN19**]. Dans [**GL21c**], on introduit deux nouvelles approches pour l'estimée L^∞ qui s'appliquent aux formes de référence semipositives non nécessairement hermitiennes (Théorème 2.3). On utilise ensuite ces estimées pour résoudre les équations de Monge-Ampère (Théorème 2.4), établissant une version hermitienne singulière du théorème de Yau (Théorème 2.6). Dans la deuxième partie de ce chapitre, on se concentre sur le volume de Monge-Ampère. En particulier, dans le Théorème 2.9 on confirme une conjecture de Demailly-Păun [**DP04**] pour certaines variétés complexes compactes non-kählériennes.

Flots de Monge-Ampère complexes

Le flot de Ricci en géométrie riemannienne a été introduit par Hamilton [**Ham82**]. C'est un flot de métriques qui évoluent selon leur tenseur de Ricci. D'après une observation de Bando, la propriété d'être kählérienne est préservée le long du flot de Ricci; le flot ainsi obtenu est appelé le flot de Kähler-Ricci. Ce dernier peut être écrit comme une équation de Monge-Ampère scalaire parabolique, dont l'existence d'une solution lisse dans un intervalle maximal est bien connue. H.D. Cao [**Cao85**] a démontré que si $c_1(X) \leq 0$ alors le flot de Kähler-Ricci existe pour tout le temps et converge en topologie C^∞ vers l'unique métrique Kähler-Einstein. Le même phénomène sur les modèles minimaux lisses de type général a été découvert par Tsuji [**Tsu88**]. Depuis, le flot de Kähler-Ricci est devenu un outil important en géométrie kählérienne.

En s'inspirant du travail célèbre de Birkar-Cascini-Hacon-Mckernan [**BCHM10**], qui montre l'existence des modèles minimaux pour les variétés de type général, Song et Tian [**ST17**] ont proposé une approche analytique analogue utilisant le flot de Kähler-Ricci. Partant d'une variété dont le diviseur canonique est nef, le flot existe pour tout temps et doit converger après une normalisation appropriée. Si le diviseur canonique n'est pas nef, le flot développe des singularités en temps fini. On espère que le flot déforme une variété en une autre plus simple à chaque temps de singularité fini, puis qu'il redémarre sur la nouvelle variété et finalement converge vers un modèle minimal.

Comme les modèles rencontrés le long du flot sont nécessairement singuliers, on est obligé de développer une théorie des solutions faibles. Une étape importante dans ce programme est de savoir démarrer le flot à partir d'un courant singulier. Dans [**ST17**], Song-Tian ont réussi à construire le flot à partir d'un courant à potentiel continu. Guedj-Zeriahi [**GZ17**] ont pu démarrer le flot à partir d'un potentiel à nombre de Lelong zero. Notre travail [**DNL17b**] permet de traiter le cas général d'un courant positif fermé de type $(1, 1)$.

Dans [**GLZ20b, GLZ20a**], les premiers éléments d'une théorie pluripotentielle parabolique ont été construits permettant de traiter le cas des singularités "Kawamata log terminales". L'idée fondamentale dans ces deux travaux est que l'on peut exprimer l'équation du flot au moyen de la théorie pluripotentielle sur la variété produit $X \times (0, T)$, qui permet de traiter arjointement la variable spatiale et la variable temporelle.

Métriques kählériennes à courbure scalaire constante

Une métrique cscK est une métrique kählérienne dont la courbure scalaire (la trace de sa forme de Ricci) est constante. C'est un cas particulier des métriques extrémales qui sont des points critiques de la fonctionnelle d'énergie de Calabi. Si la classe de cohomologie d'une métrique kählérienne est proportionnelle à la première classe de Chern $c_1(X)$, toute métrique cscK dans cette classe est Kähler-Einstein. Il y a des obstructions géométriques bien connues à l'existence des métriques cscK. La célèbre conjecture de Yau-Tian-Donaldson prédit que l'existence d'une métrique cscK est équivalente à la K-stabilité (uniforme).

Dans son travail fondateur [**Mab87**], Mabuchi a introduit une structure riemanienne sur l'espace $\mathcal{H}$ des métriques kählériennes dans une classe de cohomologie fixée $\{\omega\}$. La structure donne lieu à la notion des chemins géodésiques qui, d'après Semmes [**Sem92**] et Donaldson [**Don99**], sont solutions d'une équation de Monge-Ampère complexe homogène dégénérée. Dans le même travail, Mabuchi a introduit la fonctionnelle K-énergie que l'on notera $\mathcal{M}$, dont les points critiques sont des métriques cscK. De plus, $\mathcal{M}$ est convexe le long des géodésiques dans $\mathcal{H}$, ce qui suggère de traiter les métriques cscK comme des minimiseurs de $\mathcal{M}$. Cependant l'existence des chemins géodésiques dans $\mathcal{H}$ est problématique. On peut construire un chemin géogésique au sense faible en prenant l'enveloppe des sous-géodésiques. D'après X.X. Chen [**Che00**], Darvas-Lempert [**DL12**], Lempert-Vivas [**LV13**], et Chu-Tosatti-Weinkove [**CTW18**] la régularité optimale est $C^{1,1}$. Il est donc souhaitable de disposer de la convexité de $\mathcal{M}$ le long des géodésiques ayant cette régularité, et c'est exactement ce qu'ont démontré Berman-Berndtsson dans [**BB17**].

Dans son travail [**Che00**], Chen a montré que $\mathcal{H}$ muni de la distance de Mabuchi est un espace métrique géodésique. Une généralisation naturelle de cette métrique est la famille des métriques finslériennes de type L^p sur $\mathcal{H}$ introduites par Darvas [**Dar15**]. Ces espaces de métriques ne sont pas complets et chercher à comprendre leurs complétés est un travail important. Dans ses deux travaux [**Dar17a**], [**Dar15**], Darvas a montré que ces complétés sont des espaces d'énergie finie $\mathcal{E}^p$ étudiés précédemment par Guedj-Zeriahi [**GZ07**]. L'importance de la métrique L^1 de Darvas est que la topologie induite coincide avec la topologie forte définie précédemment par Berman-Boucksom-Eyssidieux-Guedj-Zeriahi dans [**BBE$^+$19**]. En outre, dans cette dernière topologie, les ensembles de sous-niveau de $\mathcal{M}$ sont compacts dans chaque boule fermée de $\mathcal{E}^1$.

Les travaux influents de Berman-Berndtsson [**BB17**], Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [**BBE$^+$19**], et Darvas [**Dar17a, Dar15**] ont ouvert la porte à une approche variationnelle pour l'étude des métriques cscK. Dans [**BDL17**], nous avons démontré que $\mathcal{M}$ étendue sur $\mathcal{E}^p$ est convexe et semi-continue inférieurement, et que de plus la K-énergie tordue par une forme kählérienne est strictement convexe (Théorème 4.8). On a ensuite utilisé cela dans [**BDL20**] pour démontrer que sur une variété cscK, tout minimiseur dans $\mathcal{E}^1$ de $\mathcal{M}$ est lisse (Théorème 4.13). En combinant ce résultat avec le travail de Darvas-Rubinstein [**DR17**], on peut confirmer une direction dans la conjecture de Tian reliant l'existence des métriques cscK à la coercivité de la K-énergie $\mathcal{M}$.

Une conséquence importante de notre résultat est la résolution dans une direction de la conjecture de Yau-Tian-Donaldson [**BDL20**]: une variété polarisée (X, L) admettant une métrique cscK est K-polystable. Deux ans plus tard, Chen-Cheng ont confirmé l'autre direction de cette conjecture de Tian (si $\mathcal{M}$ est coercive alors $\mathcal{H}$ contient une métrique cscK), en introduisant une équation de Monge-Ampère auxiliaire et en établissant des estimées à priori (voir [**CC21a, CC21b**]).

Une conjecture de Donaldson [**Don99**] prédit que, si $\mathcal{H}$ ne contient pas de métrique cscK et le groupe G d'automorphisme de X est trivial, alors il existe un rayon géodésique le long duquel la K-énergie est décroissante. Une version quantitative de cette conjecture a été démontrée par Chen-Cheng [**CC21b**]: il existe une métrique cscK dans $\mathcal{H}$ si et seulement si la variété est L^1-géodésiquement stable. En gros, cette dernière notion est équivalente à la strict positivité de la pente de la K-énergie le long des rayons géodésiques dans $\mathcal{E}^1$. D'après Berman-Boucksom-Jonsson [**BBJ15**], pour résoudre la version uniforme de la conjecture de Yau-Tian-Donaldson, reliant l'existence d'une métrique cscK à la K-stabilité uniforme, il faut remplacer les $\mathcal{E}^1$-géodésiques par les géodésiques provenant des configurations test. Dans [**DL20**], en utilisant l'estimée L^∞ relative entre autres, on a confirmé la version $C^{1,\bar{1}}$ de la conjecture de Donaldson (Théorème 4.18), fournissant probablement une étape intermédiare pour la résolution de la conjecture de Yau-Tian-Donaldson. Pour terminer, à l'heure actuelle, la conjecture est toujours ouverte mais elle est très activement étudiée par plusieurs communautés de recherche.

CHAPTER 1

Complex Monge-Ampère equations with prescribed singularities

We start with a brief history of the resolution of the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi}\mu,$$

on a compact Kähler manifold X of dimension n equipped with a Kähler metric ω. Here, μ is a positive Radon measure on X, $\lambda \in \mathbb{R}$ and we restrict our attention to three cases $\lambda = 0, \pm 1$. When $\mu = fdV$ with f positive and smooth, solving the above Monge-Ampère equation yields existence of Kähler metrics in the cohomology class of ω with prescribed Ricci curvature. For $\lambda \geq 0$, Aubin [**Aub78**] and Yau [**Yau78**] proved that there is a unique smooth solution obtained via a priori estimates in the continuity method. The most difficult part was known to be the L^∞-estimate in the case $\lambda \leq 0$. Yau's proof of the L^∞ estimate relies on a delicate Moser iteration process which applies to densities $f \in L^p$ for some $p > n$. Twenty years after Yau's proof, Kołodziej [**Koł98**] introduced a new approach using the Monge-Ampère capacity. His proof applies to densities $f \in L^p$ for $p > 1$ (or more generally to densities in some Orlicz space with fast growth at infinity). In relation with the Minimal Model Program in birational geometry, Kołodziej's approach has been pushed further by Eyssidieux-Guedj-Zeriahi in [**EGZ09**] where they studied Monge-Ampère equations on singular Kähler varieties. In many geometric situations one can not expect the solutions to be globally bounded, and making sense of the Monge-Ampère measure of unbounded psh functions is a delicate task. In [**GZ07**] Guedj-Zeriahi defined the non-pluripolar Monge-Ampère measure and solved degenerate Monge-Ampère equations for quite general measures on the right-hand side. This approach has been extended to the context of big cohomology classes by Boucksom-Eyssidieux-Guedj-Zeriahi [**BEGZ10**]. In [**BBGZ13**], Berman-Boucksom-Guedj-Zeriahi introduced a variational method to solve the Monge-Ampère equation without using the continuity method.

In [**DNL15, DNL17a, DNL17b**], [**DDNL18c, DDNL18b, DDNL18a, DDNL21a, DDNL21b**], [**GLZ19a**], [**BBLL19**] we obtained several results in this direction. We now present some of them in detail and introduce several open questions along the way.

1. Monotonicity of non-pluripolar Monge-Ampère mass

Let θ be a smooth closed real $(1,1)$-form on X. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (qpsh) if locally $u = \rho + \varphi$ where φ is plurisubharmonic (psh) and ρ is smooth. A qpsh function u is θ-psh if $\theta + dd^c u \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the class of all θ-psh functions on X which are not indentically $-\infty$. The cohomology class $\{\theta\}$ is big if $\mathrm{PSH}(X, \theta - \varepsilon\omega)$ is not empty for some $\varepsilon > 0$, it is nef if $\{\theta + \varepsilon\omega\}$ is Kähler for all $\varepsilon > 0$.

Throughout this section we assume that $\{\theta\}$ is big. By Demailly's regularization theorem [**Dem92, Dem94**] there exists $\psi \in \mathrm{PSH}(X, \theta - \varepsilon\omega)$ with analytic singularities, which is smooth in some Zariski open set called the ample locus of $\{\theta\}$, and denoted by $\mathrm{Amp}(\theta)$. Here, a function

$f : X \to \mathbb{R} \cup \{-\infty\}$ has analytic singularities if it can be written locally as

$$f(x) = c \log \left(\sum_{j=1}^{N} |f_j|^2 \right) + \rho,$$

where $c > 0$, ρ is smooth and the functions f_j are holomorphic.

Given two θ-psh functions u and v we say that u is more singular than v, and we write $u \preceq v$, if $u \leq v + C$ for some constant C. These two functions have the same singularities, and we write $u \simeq v$, if $u \preceq v$ and $v \preceq u$. The envelope

$$V_\theta := \sup\{u \in \mathrm{PSH}(X, \theta) \ : \ u \leq 0\}$$

is a θ-psh function with minimal singularities, it is less singular than any other θ-psh functions.

The classical Monge-Ampère capacity (see [**BT82**], [**Koł98**], [**GZ05**]) is defined by

$$\mathrm{Cap}_\omega(E) := \sup \left\{ \int_E (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X, \omega), \ -1 \leq u \leq 0 \right\}.$$

A sequence u_j converges in capacity to u if for any $\varepsilon > 0$ we have

$$\lim_{j \to +\infty} \mathrm{Cap}_\omega(\{|u_j - u| \geq \varepsilon\}) = 0.$$

A set E is quasi-open (respectively quasi-closed) if for any $\varepsilon > 0$ there exists an open (respectively closed) set U such that $\mathrm{Cap}_\omega(E \setminus U) \leq \varepsilon$ and $\mathrm{Cap}_\omega(U \setminus E) \leq \varepsilon$.

Assume $\theta_1, ..., \theta_p$, $p \leq n$, are smooth closed real $(1,1)$-forms representing big cohomology classes. Let $u_j \in \mathrm{PSH}(X, \theta_j)$, for $j = 1, ..., p$. If these functions are locally bounded in the corresponding ample loci then by Bedford-Taylor [**BT76, BT82**] we can define the Monge-Ampère product

$$(\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_p + dd^c u_p),$$

as a positive (p,p)-current on $\cap_{j=1}^n \mathrm{Amp}(\theta_j)$. Since the total mass of this current is finite, one can extend it trivially over X. As shown by Bedford-Taylor [**BT87**], the Monge-Ampère product is local in the plurifine topology: if U is a quasi-open set and $u_j = v_j$ on U for $j = 1, ..., p$, then

$$\mathbf{1}_U (\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_p + dd^c u_p) = \mathbf{1}_U (\theta_1 + dd^c v_1) \wedge ... \wedge (\theta_p + dd^c v_p).$$

We next define the non-pluripolar product following [**BEGZ10**]. For each $t > 0$, we consider $u_{j,t} := \max(u_j, V_{\theta_j} - t)$ and $U_t := \cap_{j=1}^n \{u_j > V_{\theta_j} - t\}$. By locality of the Monge-Ampère product with respect to the plurifine topology, the measures

$$\mathbf{1}_{U_t} (\theta_1 + dd^c u_{1,t}) \wedge ... \wedge (\theta_p + dd^c u_{p,t})$$

are increasing in t. The limit as $t \to +\infty$, denoted by $(\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_p + dd^c u_p)$, is a positive current on X. By Sibony [**Sib85**] (see also [**Dem12**], [**BEGZ10**]), this current is closed. In case when $u_1 = ... = u_n = u$ and $\theta_1 = ... = \theta_n = \theta$, we obtain the non-pluripolar Monge-Ampère measure of u denoted by $(\theta + dd^c u)^n$ or simply by θ_u^n. Thus, given big cohomology classes $\alpha_1, ..., \alpha_p$ one can define the positive product $\langle \alpha_1 ... \alpha_p \rangle$ as the de Rham cohomology class of the closed positive (p,p)-current:

$$(\theta_1 + dd^c V_{\theta_1}) \wedge ... \wedge (\theta_p + dd^c V_{\theta_p}),$$

where θ_j is a closed smooth real $(1,1)$-form representing α_j. As shown in [**BEGZ10**] $\langle \alpha_1 ... \alpha_p \rangle$ depends continuously on the p-tuple $(\alpha_1, ..., \alpha_p)$.

In the construction of the non-pluripolar product we somehow remove the singular part of the Monge-Ampère product. If $u \preceq v$ then the singular part corresponding to u is larger than the one corresponding to v. Having this in mind it is not surprising that the non-pluripolar mass is monotone with respect to singularity types. This is the content of the following result in [**DDNL18b**]:

THEOREM 1.1. *For each $p \in \{1, ..., n\}$, let $u_p, v_p \in \mathrm{PSH}(X, \theta_p)$ be such that $u_p \preceq v_p$. Then*

$$\int_X (\theta_1 + dd^c u_1) \wedge ... \wedge (\theta_n + dd^c u_n) \leq \int_X (\theta_1 + dd^c v_1) \wedge ... \wedge (\theta_n + dd^c v_n).$$

The result was conjectured in [**BEGZ10**] and proved by Witt-Nyström [**WN19b**] in the case when the θ_j (and u_j) are equal. A simplified proof has been recently given in [**Lu20**], [**LN20**].

2. Relative full mass classes

From Theorem 1.1 we know that the total mass of θ_u^n and θ_v^n are equal if u and v have the same singularities, but the reverse direction does not hold. For each $\phi \in \mathrm{PSH}(X, \theta)$ we define

$$\mathcal{E}(X, \theta, \phi) := \left\{ u \in \mathrm{PSH}(X, \theta) \ : \ u \preceq \phi, \ \int_X \theta_u^n = \int_X \theta_\phi^n \right\}.$$

In case $\phi = 0$ and $\theta = \omega$, we simply denote the class $\mathcal{E}(X, \omega, 0)$ by $\mathcal{E}(X, \omega)$. This is the full mass class considered in [**GZ07**]. As shown in [**GZ07**], [**CGZ08**] there are many unbounded functions in $\mathcal{E}(X, \omega)$: $-(-u)^p \in \mathcal{E}(X, \omega)$ for any $0 < p < 1$ and $u \in \mathrm{PSH}(X, \omega)$ with $u \leq -1$.

A natural question is whether there is a maximal element in $\mathcal{E}(X, \theta, \phi)$. It turns out that the existence of such a potential implies that ϕ has *model* type singularities. The latter is defined via an envelope construction due to Ross-Witt Nyström [**RWN14**] and Rashkovskii-Sigurdsson [**RS05**] that we now recall. For each Lebesgue measurable function f on X, we define

$$P_\theta(f) := \left(\sup\{ u \in \mathrm{PSH}(X, \theta) \ : \ u \leq f \text{ quasi everywhere on } X \} \right)^*.$$

Here, quasi everywhere means outside a pluripolar set, i.e. a set contained in the $-\infty$-locus of some $v \in \mathrm{PSH}(X, \omega)$. If $f = \min(u, v)$, we will simply write $P_\theta(\min(u, v)) = P_\theta(u, v)$.

DEFINITION 1.2. For a function $\phi \in \mathrm{PSH}(X, \theta)$ we define

$$P_\theta[\phi] := \left(\lim_{t \to +\infty} P_\theta(\min(\phi + t, 0)) \right)^*.$$

A function $\phi \in \mathrm{PSH}(X, \theta)$ is a model potential if $P_\theta[\phi] = \phi$ and $\int_X \theta_\phi^n > 0$.

If f is lower semicontinuous, by a balayage process one can show that the Monge-Ampère measure $(\theta + dd^c P_\theta(f))^n$ is supported on the contact set $\{P_\theta(f) = f\}$. For more general obstacles f, we proved the following in [**GLZ19a**]:

LEMMA 1.3. *If f is quasi lower semicontinuous on X and $P_\theta(f) \in \mathrm{PSH}(X, \theta)$, then $(\theta + dd^c P_\theta(f))^n$ is supported on the contact set $\{P_\theta(f) = f\}$. In particular, if $f = \min(u, v)$ with $u, v \in \mathrm{PSH}(X, \theta)$, then*

$$(\theta + dd^c P_\theta(u, v))^n \leq \mathbf{1}_{\{P_\theta(u,v) = u\}} \theta_u^n + \mathbf{1}_{\{P_\theta(u,v) = v\}} \theta_v^n.$$

Here, we say that f is quasi lower semicontinuous if for each $\varepsilon > 0$ there exists an open set U of small capacity $\mathrm{Cap}_\omega(U) < \varepsilon$, such that $f|_{X \setminus U}$ is lower semicontinuous. If f is quasi lsc, there exists a decreasing sequence of lsc functions converging to f quasi everywhere. We also stress that the regularity of f in Lemma 1.3 is necessary. For a counter example, we take f to be 0 in some small open ball B and $+\infty$ on $X \setminus B$. If $(\theta + dd^c P_\theta(f))^n$ is supported in the contact set $D = \{P_\theta(f) = f\} \subset B$ then $\int_X (\theta + dd^c P_\theta(f))^n \leq \int_B \theta^n$. But $P_\theta(f) \simeq V_\theta$, so the total mass $\int_X (\theta + dd^c P_\theta(f))^n$ must be the volume of $\{\theta\}$.

An important consequence of Lemma 1.3 is that if u, v are two supersolutions:

$$(\theta + dd^c u)^n \leq e^u \mu \ , \ (\theta + dd^c v)^n \leq e^v \mu,$$

then the envelope $P_\theta(u,v)$ is also a supersolution. Building on this idea we initiated in [**GLZ19a**] a supersolution method to solve complex Monge-Ampère equations by taking the envelope of supersolutions:

$$P_\theta(\inf\{u \text{ supersolution}\}).$$

An important tool in pluripotential theory is the local comparison principle of Bedford-Taylor [**BT82**] for bounded psh functions. The global version for the class $\mathcal{E}(X,\omega)$ was given by Guedj-Zeriahi in [**GZ07**]. As a consequence of Theorem 1.1 we proved in [**DDNL18b**] that it also holds for relative full mass classes.

PROPOSITION 1.4 (Comparison principle). *If $u, v \in \mathcal{E}(X,\theta,\phi)$ then*

$$\int_{\{v<u\}} (\theta + dd^c u)^n \leq \int_{\{v<u\}} (\theta + dd^c v)^n.$$

From the comparison principle and the resolution of the Monge-Ampère equation we obtained the following domination principle in [**DDNL18b**] which is a crucial tool for later developments.

LEMMA 1.5 (Domination principle). *Assume $u, v \in \mathcal{E}(X,\theta,\phi)$ and $\phi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\phi^n > 0$.*

(1) If $\mathbf{1}_{\{u<v\}}\theta_u^n \leq c\mathbf{1}_{\{u<v\}}\theta_v^n$ for some $0 \leq c < 1$, then $u \geq v$.

(2) In particular, if $\theta_u^n \geq e^{u-v}\theta_v^n$ then $u \leq v$.

If θ is semipositive then $V_\theta = 0$ and the Monge-Ampère measure $(\theta + dd^c V_\theta)^n = \theta^n$ is very regular. In an influential work [**Ber19**] Berman introduced a regularization process for the envelope V_θ by solving a family of Aubin-Yau equations parametrized by large $\beta > 0$:

$$\left(\theta + \frac{1}{\beta}\omega + dd^c u_\beta\right)^n = e^{\beta u_\beta} dV. \tag{1.1}$$

There are several applications of Berman's β-convergence method. We prove in [**LN15**] that singular (ω, m)-subharmonic functions on compact Kähler manifolds can be smoothly approximated. Chu-Zhou [**CZ19**], Tosatti [**Tos18**] independently prove optimal regularity of the Monge-Ampère envelope by following Berman's scheme. We show in [**BL18**] that the twisted Kähler-Ricci flow similarly parametrized as above converges to a flow of moving free boundaries. In [**GLZ19a**] we use Berman's method to show that the psh envelope of a viscosity supersolution is a pluripotential supersolution, emphasizing an intimate relation between the two theories.

In case θ is additionally nef, the solutions u_β to (1.1) are smooth and by establishing a uniform Laplacian estimate locally in the ample locus of $\{\theta\}$, Berman showed that V_θ is $C^{1,\bar{1}}_{\mathrm{loc}}$ in $\mathrm{Amp}(\{\theta\})$. In particular, $(\theta + dd^c V_\theta)^n = \mathbf{1}_{\{V_\theta=0\}}\theta^n$. In the same work Berman showed that for a general big class the inequality $(\theta + dd^c V_\theta)^n \leq \mathbf{1}_{\{V_\theta=0\}}\theta^n$ holds.

We proved in [**DDNL18b**] that the same property holds if we replace V_θ by any model potential.

LEMMA 1.6. *If $u \in \mathrm{PSH}(X,\theta)$ then $(\theta + dd^c P_\theta[u])^n \leq \mathbf{1}_{\{P_\theta[u]=0\}}\theta^n$.*

In fact, it was recently proved in [**DNT19**] that the inequality above is actually an equality. It is natural to ask the following question:

QUESTION 1. *Assume $u \leq v$ are bounded ω-psh functions. Under what condition do we have*

$$\mathbf{1}_{\{u=v\}}(\omega + dd^c u)^n = \mathbf{1}_{\{u=v\}}(\omega + dd^c v)^n?$$

There are counterexamples to the equality above when $(\omega + dd^c v)^n$ does not have a density. The expectation is thus that it holds when $(\omega + dd^c v)^n$ is absolutely continuous with respect to Lebesgue measure.

A direct consequence of the above analysis and the domination principle is the following property of model potentials:

PROPOSITION 1.7. *If $u \in \mathrm{PSH}(X,\theta)$ and $\int_X \theta_u^n > 0$ then $\phi := P_\theta[u]$ is a model potential. If ϕ is a model potential and $u \in \mathrm{PSH}(X,\theta)$ with $u \preceq \phi$, then $u - \sup_X u \leq \phi$.*

In particular, if $\int_X \theta_u^n > 0$ then $P[P[u]] = P[u]$. We are not aware of any example showing that the equality does not hold.

CONJECTURE 1.8. *For any $u \in \mathrm{PSH}(X,\theta)$ we have $P_\theta[P_\theta[u]] = P_\theta[u]$.*

An important consequence of the monotonicity theorem is that if $\{\theta\}$ is nef, the potentials in $\mathcal{E}(X,\theta)$ have zero Lelong number everywhere. This is one of our main results in [**DDNL18c**], positively answering an open question in [**DGZ16**]:

THEOREM 1.9. *Assume $\{\theta\}$ is a big and nef class. If $u \in \mathcal{E}(X,\theta)$ and $\theta \leq \omega$, then $u \in \mathcal{E}(X,\omega)$. In particular, the Lelong number of u is zero everywhere on X.*

The vanishing of the Lelong numbers of functions in $\mathcal{E}(X,\theta)$ was proved in [**GZ07**] when θ is Kähler and in [**BBE$^+$19**] when θ is the pull back of a Kähler form on a normal Kähler variety via a resolution of singularities.

3. Relative L^∞ estimates

We consider the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = f dV,$$

where $0 \leq f \in L^1(X)$ is such that $\int_X f dV = \int_X \omega^n$. It follows from [**GZ07**] and [**Din09**] that there exists a unique normalized solution $\varphi \in \mathcal{E}(X,\omega)$. If f belongs to $L^p(X)$ for some $p > 1$ (or some Orlicz class with fast growth at infinity), it follows from Kołodziej's work [**Koł98**] that φ is bounded, and continuous. His approach relies on the Monge-Ampère capacity and Bedford-Taylor's comparison principle. This technique has been further generalized in [**EGZ09, EGZ08, DP10, BEGZ10**] in order to deal with less positive or collapsing classes.

It is very natural to look for a similar result when the density f is merely smooth and positive on the complement of some divisor D but not in L^p for any $p > 1$. In such a situation, a reasonable expectation is that φ is locally bounded away from D with a uniform relative L^∞-estimate of the form $\varphi \geq \psi - C$, where ψ is some given quasi-psh function. In [**DNL17a**] we proved that this type of estimate does hold when the blow up behaviour of f is controlled by $e^{|\psi|}$, for some quasi-psh function ψ.

THEOREM 1.10. *Assume $f \leq e^{-\psi}$ for some quasi-psh function ψ. Then, for each $a \in (0,1)$,*

$$\varphi - \sup_X \varphi \geq a\psi - C,$$

where C is a constant depending on upper bound for p, $\int_X e^{-2u/a}$, a^{-1}.

It follows from [**GZ07**] and Skoda's integrability theorem [**Sko72**], [**Zer01**], that $\int_X e^{-2u/a} dV$ is uniformly bounded.

The proof uses the generalized Monge-Ampère capacity

$$\mathrm{Cap}_{\theta,\psi}(E) := \sup\left\{\int_E (\theta + dd^c v)^n \ : \ v \in \mathrm{PSH}(X,\omega),\ \psi - 1 \leq v \leq \psi\right\},$$

which was further investigated in [**DNL15**]. In [**Lu20**], we proved that all these capacities are uniformly comparable, yielding a simple proof of the integration by parts formula, an essential tool in the variational approach to solve complex Monge-Ampère equations.

Building on this relative L^∞ estimate we proved in [**DNL17a**] that the solution φ is smooth in $X \setminus D$ when f is smooth there and can be written as $f = e^{\psi^+ - \psi^-}$, for some quasi-psh functions $\psi^\pm$. Of course, we expect that the latter condition can be removed:

Question 2. *Assume $f \in C^\infty(X \setminus D)$ and $f \in L^1(X)$. Do we have $\varphi \in C^\infty(X \setminus D)$?*

By a similar idea we prove in [**DDNL21a**] the following relative estimate involving functions in relative full mass classes.

Theorem 1.11. *Fix $a \in (0,1)$, $A > 0$, $\psi \in \mathrm{PSH}(X, \theta)$, and $f \in L^p(X, dV)$, $p > 1$. Assume that*

$$\int_E f dV \leq A(\mathrm{Cap}_{\theta,\psi}(E))^2, \tag{1.2}$$

for all Borel sets $E \subset X$. If $u \in \mathrm{PSH}(X,\theta)$ verifies $\psi \preceq P_\theta[u]$ and

$$(\theta + dd^c u)^n \leq a(\theta + dd^c \psi)^n + f dV, \tag{1.3}$$

then for a uniform constant $C = C(\|f\|_p, p, (1-a)^{-1}, A)$ we have

$$u - \sup_X u \geq \psi - \sup_X \psi - C.$$

As shown in [**DDN21**], there is an alternative proof of the L^∞ estimate in the breakthrough articles of Chen-Cheng [**CC21a, CC21b**] relying on our relative estimate above.

The condition (1.2) is satisfied when e.g. ψ is a model potential or ψ is strictly θ-psh. In (1.3) the assumption $a \in (0,1)$ is essential for our proof. Whether one can take $a = 1$ is a widely open question even in the case when ψ is bounded.

Question 3 (Kołodziej). *Assume ψ is a bounded ω-psh function and $u \in \mathcal{E}(X,\omega)$ satisfies*

$$(\omega + dd^c u)^n \leq C(\omega + dd^c \psi)^n,$$

for some constant $C > 0$. Is u also bounded?

4. Solving Monge-Ampère equations

In this section we focus on the resolution of complex Monge-Ampère equations

$$(\theta + dd^c \varphi)^n = e^{\lambda\varphi}\mu, \ \varphi \in \mathcal{E}(X, \theta, \phi), \tag{MA}$$

where μ is a non-pluripolar positive Radon measure on X, ϕ is a model potential and $\lambda \geq 0$. When $\lambda = 0$, a necessary condition for existence of solutions is that $\mu(X) = \int_X (\theta + dd^c\phi)^n$. When the model potential $\phi = V_\theta$ has minimal singularities, it was shown in [**GZ07, BEGZ10, BBGZ13**] that there exists a solution, which turns out to be unique by Dinew's work [**Din09**]. In [**DDNL18b, DDNL21a**] we generalized these results for arbitrary model potential ϕ.

Theorem 1.12. *The equation* (MA) *admits a unique solution.*

In case $\lambda > 0$, the uniqueness follows from the domination principle. When $\lambda = 0$, the uniqueness is understood in the following sense: if u and v are two solutions then $u - v$ is constant. This can be proved by following Dinew's argument [**Din09**]. A new proof using quasi-psh envelopes has been provided in [**LN20**].

In case μ has finite energy, i.e. $\mathcal{E}^1(X,\theta,\phi) \subset L^1(\mu)$, it can be shown that the solution ϕ belongs to the finite energy class $\mathcal{E}^1(X,\theta,\phi)$ defined by

$$\mathcal{E}^1(X,\theta,\phi) := \left\{ u \in \mathcal{E}(X,\theta,\phi) \ : \ \int_X (u - \phi)\theta_u^n > -\infty \right\}.$$

We proved this result in [**BBLL19**] when $\theta = \omega$ is Kähler, and applied it to give a large deviation principle in the setting of weighted pluripotential theory arising from polynomials associated to a convex body P in $(\mathbb{R}^+)^d$.

We briefly describe the proof of the existence part for $\lambda = 0$, $\mu = fdV$ with $f \in L^p(X, dV)$, $p > 1$, which is inspired by the supersolution method introduced in [**GLZ19a**]. For each $a \in (0,1)$, $t > 0$, using [**BEGZ10**] we solve

$$(\theta + dd^c\varphi_{a,t})^n = c(a)\mu + a\mathbf{1}_{\{\phi \leq V_\theta - t\}}(\theta + dd^c \max(\phi, V_\theta - t))^n,$$

with $\varphi_{a,t} \in \mathcal{E}(X,\theta)$, $\sup_X \varphi_{a,t} = 0$. Here, $c(a) > 0$ is a normalization constant ensuring that the total mass of the LHS measure is $\mathrm{Vol}(\theta)$. It is not hard to see that $c(a) \to 1$ as $a \to 1$. We next want to take the envelope $P_\theta(\varphi_{a,j}, \varphi_{a,j+1}....)$. To ensure that it is not identically $-\infty$, we need a relative L^∞ estimate of type

$$\varphi_{a,t} \geq \phi - C,$$

where C does not depend on t. This is provided by Theorem 1.11 since $a < 1$. Using Lemma 1.3 we can then show that the function $\varphi_a := (\lim_{j\to+\infty} P_\theta(\varphi_{a,j}, u_{\varphi,j+1}, ...))^*$ verifies

$$\phi - C \leq \varphi_a, \text{ and } (\theta + dd^c\varphi_a)^n \leq c(a)\mu.$$

We next use the same envelope construction and let $a \to 1$ to get $\varphi \in \mathrm{PSH}(X,\theta)$, with $\phi \preceq \varphi$ and

$$(\theta + dd^c\varphi)^n \leq \mu.$$

We finally use the monotonicity of mass to conclude.

When $\mu = fdV$ for some $f \in L^p$, $p > 1$, our relative estimate shows that the solution φ has the same singularities as ϕ. In case f is smooth it is expected that the solution φ is smooth where ϕ is. But even in the case $\phi = V_\theta$, this is a widely open question.

QUESTION 4. *Assume ϕ is quite regular (e.g. smooth/continuous) in some Zariski open set U and $f > 0$ is regular (smooth/continuous) on X. Is φ smooth/continuous in U?*

5. Log-concavity of volume

For a closed positive $(1,1)$-current T, one can find a closed smooth real $(1,1)$-form θ and a function $u \in \mathrm{PSH}(X,\theta)$ such that $T = \theta + dd^c u$. The (non-pluripolar) volume of T is defined as

$$\mathrm{Vol}(T) := \int_X (\theta + dd^c u)^n.$$

If $\{\theta\}$ is big then the non-pluripolar Monge-Ampère measure $(\theta + dd^c u)^n$ is well-defined, hence so is $\mathrm{Vol}(T)$. Otherwise, we simply set $\mathrm{Vol}(T) = 0$. By solving Monge-Ampère equations with prescribed singularities, in [**DDNL21a**] we confirmed a conjecture of Boucksom-Eyssidieux-Guedj-Zeriahi [**BEGZ10**]:

THEOREM 1.13. *The function $T \mapsto \log \int_X T^n$, defined on the cone of closed positive $(1,1)$-currents, is concave.*

By an elementary argument the proof is reduced to showing that

$$\int_X T_1 \wedge .... \wedge T_n \geq \prod_{j=1}^n \mathrm{Vol}(T_j)^{1/n},$$

if $T_j = \theta_j + dd^c u_j \geq 0$ with $\{\theta_j\}$ big and $u_j \in \mathrm{PSH}(X,\theta_j)$. If one of the T_j has zero mass, the right-hand side will be zero and we are done. We can thus fix, pour each j, a model potential $\phi_j = P_{\theta_j}[u_j] \in \mathrm{PSH}(X,\theta_j)$. We fix a probability volume form dV and solve

$$(\theta_j + dd^c v_j)^n = c_j dV, \ v_j \in \mathcal{E}(X,\theta_j,\phi_j),$$

where $c_j = \int_X (\theta_j + dd^c\phi_j)^n = \mathrm{Vol}(T_j)$. By the mixed Monge-Ampère inequality, we have

$$(\theta_j + dd^c v_1) \wedge ... \wedge (\theta_n + dd^c v_n) \geq (c_1...c_n)^{1/n} dV.$$

Integrating this over X we obtained the result.

The key of this proof is the resolution of the complex Monge-Ampère equation with prescribed singularities obtained in [**DDNL21a**]. One can also solve this equation by a variational method, introduced in [**BBGZ13**], as explained in [**DDNL18b**], [**Xia19**], [**Lu20**].

CHAPTER 2

Pluripotential theory on compact Hermitian manifolds

In this chapter, we present recent joint works with T.T. Phung and T.D. Tô [**LPT20**], and V. Guedj [**GL21a, GL21b, GL21c**], and discuss several open questions for possible future projects.

1. Complex Monge-Ampère equations

The Monge-Ampère equation on compact complex manifolds was first studied in the eighties by Cherrier [**Che87**] who tried to solve the Hermitian analogue of Yau's theorem. He succeeded in giving a priori C^2 estimates, but only provided an L^∞ estimate under a restrictive condition. After several attempts by Hanani [**Han96**], Guan-Li [**GL10**], the missing L^∞ estimate has been established by Tosatti-Weinkove [**TW10**], who solved in full generality the following analogue of Yau's theorem:

THEOREM 2.1. [**TW10**] *Let (X, ω_X) be a compact Hermitian manifold of dimension n and $0 < f$ is a smooth function on X. Then there exists a unique couple φ, c such that $c > 0$ and $\omega + dd^c\varphi > 0$, solving*

$$(\omega_X + dd^c\varphi)^n = cf\omega_X^n.$$

In contrast with the Kähler case, here the mysterious constant $c = c_f$ depends heavily on f. We have shown in [**LPT20**] that c_f varies continuously in f:

PROPOSITION 2.2. *Assume that $0 \leq f, g \in L^p(X)$ for some $p > 1$. Then*

$$|c_f - c_g| \leq C\|f - g\|_p^{1/n},$$

where $C > 0$ is a constant depending on p, an upper bound for $\|f\|_p, \|g\|_p$ and a positive lower bound for $\|f\|_{1/n}$, $\|g\|_{1/n}$.

After the appearance of [**TW10**], the study of complex Monge-Ampère equations on compact Hermitian manifolds has gained considerable interest. The smooth Gauduchon-Calabi-Yau conjecture has been further solved by Székelyhidi-Tosatti-Weinkove [**STW17**], while the pluripotential theory has been partially extended by Dinew, Kołodziej, and Nguyen [**DK12, KN15, Din16, KN19**].

The main difficulty in establishing the L^∞-estimate in [**TW10**] and [**DK12**] lies in the fact that the reference form is not closed, which produces many extra terms to handle when using Stokes theorem. It is also highly non trivial to get uniform bounds on the total Monge-Ampère volumes involved in the estimates. Błocki has provided a different approach [**Bło05, Bło11**] which is based on the Alexandroff-Bakelman-Pucci maximum principle and a stability estimate due to Cheng-Yau (L^2-case) and Kołodziej (L^p-case). Błocki's method works in the Hermitian case and has been generalized to various settings by Székelehydi, Tosatti and Weinkove [**STW17, Szé18, TW18**].

In [**GL21c**] we have provided a new and direct alternative proof of this a priori estimate, relying only on the local resolution of the classical Dirichlet problem for the complex Monge-Ampère equation, and twisting the right hand side with an exponential.

All the above methods require the reference form to be strictly positive. In studying Yau's analogue on singular varieties, one is lead to consider forms which are merely semipositives.

Given a semi-positive form ω, we introduce several positivity properties:

- we say ω is *non-collapsing* if there is no bounded ω-plurisubharmonic function u such that $(\omega + dd^c u)^n \equiv 0$;
- we say ω is *uniformly non-collapsing* if

$$v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X, \omega) \cap L^\infty(X) \right\} > 0.$$

In the same work [**GL21c**] we have also introduced yet another new approach for establishing uniform a priori estimates, which applies in the context of semipositive forms once the Monge-Ampère volume $v_-(\omega)$ is under control:

THEOREM 2.3. *Let ω be semi-positive and uniformly non-collapsing. Let μ be a probability measure such that $\mathrm{PSH}(X, \omega) \subset L^m(\mu)$ for some $m > n$. Any solution $\varphi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ to*

$$(\omega + dd^c \varphi)^n = c\mu,$$

where $c > 0$, satisfies

$$\mathrm{Osc}_X(\varphi) \leq T$$

for some uniform constant T which depends on an upper bound on $\frac{c}{v_-(\omega)}$ and

$$A_m(\mu) := \sup \left\{ \left(\int_X (-\psi)^m d\mu \right)^{\frac{1}{m}} \ : \ \psi \in \mathrm{PSH}(X, \omega) \ \textit{with} \ \sup_X \psi = 0 \right\}.$$

While the pluripotential approach consists in measuring the Monge-Ampère capacity of sublevel sets $(\varphi < -t)$, we directly measure the volume of the latter, avoiding delicate integration by parts. This result covers the case when $\mu = f dV_X$ is absolutely continuous with respect to Lebesgue measure, with density f belonging to L^p, $p > 1$, or to an appropriate Orlicz class L^w (for some convex weight w with "fast growth" at infinity), thus partially extending the case of Hermitian forms treated by Dinew-Kołodziej [**DK12**] and Kołodziej-Nguyen [**KN15, KN20**].

The non-collapsing condition is the minimal positivity condition one should require. We showed in [**GL21b**] that it implies the *domination principle*, a useful extension of the classical maximum principle. We also provided a simple example showing that having positive volume $\int_X \omega^n > 0$ does not prevent from being collapsing. We also showed in [**GL21b**] that ω is uniformly non-collapsing if one restricts to ω-psh functions that are uniformly bounded by a fixed constant M:

$$v_M^-(\omega) := \inf \left\{ \int_X (\omega + dd^c u)^n \ : \ u \in \mathrm{PSH}(X, \omega) \ \text{with} \ -M \leq u \leq 0 \right\} > 0.$$

The proof relies exclusively on the quasi-psh envelope. Given this it is natural to ask

QUESTION 5. *Do we have an explicit lower bound for $v_M^-(\omega)$ in terms of M?*

We next focus on solving the complex Monge-Ampère equations with a semi-positive reference form ω. By analogy with the Kähler setting, we say that ω is *big* if there exists an ω-psh function with analytic singularities ρ such that $\omega + dd^c \rho \geq \delta \omega_X$ for some $\delta > 0$.

THEOREM 2.4. *Let ω be a semi-positive $(1,1)$ form which is either big or such that $v_-(\omega) > 0$. Fix $0 \leq f \in L^p(dV_X)$, where $p > 1$ and $\int_X f dV_X = 1$. Then*

- *there exists a unique constant $c(\omega, f) > 0$ and a bounded ω-psh function φ such that $(\omega + dd^c \varphi)^n = c(\omega, f) f dV_X$;*
- *for any $\lambda > 0$ there exists a unique $\varphi_\lambda \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ such that*

$$(\omega + dd^c \varphi_\lambda)^n = e^{\lambda \varphi_\lambda} f dV_X.$$

When ω is Hermitian, this result has been proved by Kołodziej-Nguyen [**KN15**] and Nguyen [**Ngu16**], who also proved uniqueness of the solution for $\lambda > 0$.

We next apply our results to solve a singular version of the Hermitian Calabi-Yau theorem. Let V be a compact complex variety with *log-terminal singularities*, i.e. V is a normal complex space such that the canonical bundle K_V is $\mathbb{Q}$-Cartier and for some (equivalently any) resolution of singularities $\pi : X \to V$, we have

$$K_X = \pi^* K_V + \sum_i a_i E_i,$$

where the E_i's are exceptional divisors with simple normal crossings, and the rational coefficients a_i (the discrepancies) satisfy $a_i > -1$.

Given ϕ a smooth metric of K_V and σ a non vanishing local holomorphic section of $K_V^{\otimes r}$, we consider the "adapted volume form"

$$\mu_\phi := \left(\frac{i^{rn^2} \sigma \wedge \overline{\sigma}}{|\sigma|^2_{r\phi}} \right)^{\frac{1}{r}}.$$

This measure is independent of the choice of σ, and it has finite mass on V, since the singularities are log-terminal. Given ω_V a Hermitian form on V, there exists a unique metric $\phi = \phi(\omega_V)$ on K_V such that

$$\omega_V^n = \mu_\phi.$$

Definition 2.5. The Ricci curvature form of ω_V is $\mathrm{Ric}(\omega_V) := -dd^c \phi$.

Recall that the Bott-Chern space $H^{1,1}_{BC}(V, \mathbb{R})$ is the space of closed real $(1,1)$-forms modulo the image of dd^c acting on real functions. The form $\mathrm{Ric}(\omega_V)$ determines a class $c_1^{BC}(V)$ which maps to the usual Chern class $c_1(V)$ under the natural surjection $H^{1,1}_{BC}(V, \mathbb{R}) \to H^{1,1}(V, \mathbb{R})$.

By analogy with the Calabi conjecture from Kähler geometry, it is natural to wonder whether conversely any representative $\eta \in c_1^{BC}(V)$ can be realised as the Ricci curvature form of a Hermitian metric ω_V. In [**GL21c**] we provided a positive answer:

Theorem 2.6. *Let V be a compact Hermitian variety with log terminal singularities equipped with a Hermitian form ω_V. For every smooth closed real $(1,1)$-form η in $c_1^{BC}(V)$, there exists a function $\varphi \in \mathrm{PSH}(V, \omega_V)$ such that*

- *φ is globally bounded on V and smooth in V_{reg};*
- *$\omega_V + dd^c\varphi$ is a Hermitian form and $\mathrm{Ric}(\omega_V + dd^c\varphi) = \eta$ in V_{reg}.*

In particular if $c_1^{BC}(V) = 0$, any Hermitian form ω_V is "dd^c-cohomologous" to a Ricci flat Hermitian current. Understanding the asymptotic behavior of these singular Ricci flat currents near the singularities of V is, as in the Kähler case, an important open problem.

Passing to a resolution of singularities $\pi : X \to V$, the equation

$$\mathrm{Ric}(\omega_V + dd^c\varphi) = \eta$$

is translated into the following complex Monge-Ampère equation

$$(\omega + dd^c u)^n = f dV_X,$$

where $\omega := \pi^*\omega_V$ is a semipositive smooth $(1,1)$-form which is big, and $f = e^{\psi^+ - \psi^-}$ has poles and zeros along a simple normal crossing divisor. The existence of φ in Theorem 2.6 is thus a consequence of Theorem 2.4, while the smoothness is obtained via a C^2-estimate building on works of [**GL10**], [**TW18**]. The uniqueness of u is known only when the density f is strictly positive, and is a consequence of our stability result in [**GL21c**] and [**LPT20**]:

THEOREM 2.7. *Assume ω is either big or uniformly non-collapsing. Fix $f_1, f_2 \in L^p(dV_X)$ with $p > 1$ and $A^{-1} \leq \left(\int_X f_i^{\frac{1}{n}} dV_X\right)^n \leq \left(\int_X f_i^p dV_X\right)^{\frac{1}{p}} \leq A$, for some constant $A > 1$. Assume $\varphi_1, \varphi_2 \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ satisfy*

$$(\omega + dd^c\varphi_i)^n = e^{\lambda\varphi_i} f_i dV_X.$$

(1) *If $\lambda > 0$, then $||\varphi_1 - \varphi_2||_\infty \leq T||f_1 - f_2||^{\frac{1}{n}}$, where T is a constant which depends on n, p and upper bounds for A, λ^{-1}, λ.*
(2) *If $\lambda = 0$ and $f_1 \geq c_0 > 0$, then the same estimate holds with T depending on n, p and upper bounds for A, c_0^{-1}.*

In particular, there is at most one bounded ω-psh solution φ to the equation $(\omega + dd^c\varphi)^n = e^{\lambda\varphi} f\, dV_X$ in case $\lambda > 0$, or when $\lambda = 0$ and $f \geq c_0 > 0$. The idea of the proof of (1) comes from [**GLZ18**] where we used a perturbation argument going back to Kołodziej [**Koł96**].

When ω is Hermitian such a stability estimate has been provided by Kołodziej-Nguyen for $\lambda = 0$ [**KN19**]. To prove Theorem 2.7 we adapt some arguments of [**LPT20**], [**GLZ18**], who treated the case $\lambda > 0$, and obtained refined estimates in the case $\lambda = 0$. When ω is Hermitian, using the stability estimates above we showed in [**LPT20**] that the solution φ is Hölder continuous with an exponent as good as in the Kähler case obtained in [**DDG$^+$14**].

THEOREM 2.8. *Fix $0 \leq f \in L^p(X), p > 1$ with $\int_X f dV_X > 0$. Then any solution u to $(\omega_X + dd^c u)^n = c_f f dV_X$ is Hölder continuous with Hölder exponent in $(0, p_n)$, where $p_n = \frac{2}{nq+1}$.*

This result has been proved in [**KN19**] for $f \geq c_0 > 0$ strictly positive and with a less precise exponent. The new input of our approach is that we use the stability estimate for $\lambda = 1$ which allows to avoid the strict positivity assumption on f and to improve the Hölder exponent.

2. Bounds on Monge-Ampère volumes

Bounding from below $v_-(\omega)$ is a very delicate issue. When ω is closed, simple integration by parts reveals that $v_-(\omega) = \int_X \omega^n$ is positive as soon as ω is positive at some point; Demailly and Păun showed in [**DP04**] that in this case ω contains a Kähler current. In the same paper, they have also proposed the following conjecture (see [**DP04**, Conjecture 0.8]): if a nef class $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ satisfies $\alpha^n > 0$, then it should contain a Kähler current, i.e. a positive closed $(1,1)$-current which dominates a Hermitian form. Recall that the Bott-Chern cohomology group $H^{1,1}_{BC}(X, \mathbb{R})$ is the quotient of closed real smooth $(1,1)$-forms, by the image of $\mathcal{C}^\infty(X, \mathbb{R})$ under the dd^c-operator. Demailly-Păun's conjecture is a transcendental version of a conjecture of Grauert-Riemenschneider [**GR70**] who asked whether the existence of a semi-positive holomorphic line bundle $L \to X$ with $c_1(L)^n > 0$ implies that X is Moishezon (i.e. bimeromorphically equivalent to a projective manifold). This conjecture has been solved positively by Siu in [**Siu84**] (with complements by [**Siu85**] and Demailly [**Dem87**]).

This influential conjecture has been further reinforced by Boucksom-Demailly-Păun-Peternell who proposed a weak transcendental form of Demailly's holomorphic Morse inequalities [**BDPP13**, Conjecture 10.1]. This stronger conjecture has been solved recently by Witt-Nyström when X is projective [**WN19a**].

Building on works of Chiose [**Chi16a**], Xiao [**Xia15**] and Popovici [**Pop16**] we have obtained in [**GL21b**] the following answer to the qualitative part of these conjectures:

THEOREM 2.9. *Let $\alpha, \beta \in H^{1,1}_{BC}(X, \mathbb{C})$ be nef classes such that $\alpha^n > n\alpha^{n-1} \cdot \beta$. The following properties are equivalent:*

(1) *$\alpha - \beta$ contains a Kähler current;*
(2) *$v_+(\omega_X) < +\infty$;*

(3) *X belongs to the Fujiki class.*

The Fujiki class is the class of compact complex manifolds that are bimeromorphically equivalent to Kähler manifolds.

Here

$$v_+(\omega_X) := \sup \left\{ \int_X (\omega_X + dd^c\varphi)^n \ : \ \varphi \in \mathrm{PSH}(X,\omega_X) \cap L^\infty(X) \right\}.$$

Building on works of Chiose [**Chi16b**] and Guan-Li [**GL10**] we have provided several results which ensure that the condition $v_+(\omega_X) < +\infty$ is satisfied:

- for any compact complex manifold X of dimension $n \leq 2$;
- for any threefold which admits a pluriclosed metric $dd^c\tilde{\omega}_X = 0$;
- as soon as there exists a metric $\tilde{\omega}_X$ such that $dd^c\tilde{\omega}_X = 0$ and $dd^c\tilde{\omega}_X^2 = 0$;
- as soon as X belongs to the Fujiki class $\mathcal{C}$.

More generally, we have proved in [**GL21b**] the following:

THEOREM 2.10. *The condition $v_+(\omega_X) < +\infty$ is independent of the choice of ω_X; it is moreover invariant under bimeromorphic transformation of compact complex manifolds.*

The condition $v_-(\omega_X) > 0$ is also independent of the choice of ω_X and invariant under bimeromorphic transformation of compact complex manifolds.

In particular these conditions both hold true if X belongs to the Fujiki class.

We are not aware of a single example of a compact complex manifold such that $v_+(\omega_X) = +\infty$ or $v_-(\omega_X) = 0$. This is an important open problem.

QUESTION 6. *Does there exist a compact Hermitian manifold X such that $v_-(\omega_X) = 0$ or $v_+(\omega_X) = +\infty$?*

QUESTION 7. *If X, Y are two compact complex manifolds which satisfy $v_-(\omega_X) > 0$ and $v_-(\omega_Y) > 0$ (respectively $v_+(\omega_X) < +\infty$ and $v_+(\omega_Y) < +\infty$), does the same property holds for $X \times Y$?*

The proof of Theorem 2.10 relies on a fine use of quasi-plurisubharmonic envelopes, which have been systematically studied in [**GLZ19a**] in the Kähler framework.

A consequence of our analysis is that the conjectures of Demailly-Păun and Boucksom-Demailly-Păun-Peternell can be extended to non closed forms, so that it makes sense outside the Fujiki class. Progresses in the theory of complex Monge-Ampère equations on compact Hermitian manifolds have indeed shown that it is useful to consider dd^c-perturbations of non closed nef forms.

As we have seen the non-pluripolar Monge-Ampère measure has found many impressive applications. It is natural to define similar notions in the Hermitian setting. The situation is however quite complicated. One may try to define

$$\mathrm{MA}(u) := \lim_{t \to +\infty} \mathbf{1}_{\{u > -t\}} (\omega + dd^c \max(u, -t))^n,$$

having in mind that the family of measures is increasing in t. But it is not clear whether the total mass is bounded. Another difficulty we have to deal with is the vanishing of the Monge-Ampère constant. Given $0 \leq f \in L^1(X, dV)$ we can define $c(\omega, f) := \lim_{j \to +\infty} c(\omega, \min(f, j)) \geq 0$.

QUESTION 8. *Does there exist $0 \leq f \in L^1(X, dV)$ with $c(\omega, f) = 0$?*

CHAPTER 3

Parabolic complex Monge-Ampère equations

1. Introduction

The Ricci flow in Riemannian geometry was first introduced by Hamilton [**Ham82**]. It is a geometric flow evolving a Riemannian metric by its Ricci curvature. As observed by Bando, starting from a Kähler metric the Ricci flow remains Kähler and it is called the Kähler-Ricci flow. The main point is that the flow can be written as a parabolic complex Monge-Ampère equation.

After the spectacular use of the Ricci flow by Perelman to settle the Poincaré and Geometrization conjectures, it is expected that the Kähler-Ricci flow can be used similarly to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time.

The classification of complex projective surfaces was established by the Italian school in the 19th century. If X is a smooth complex projective surface of non-negative Kodaira dimension then there exists a smooth projective manifold Y birational to X such that the canonical bundle of Y is nef. If the Kodaira dimension of X is $-\infty$ then X is birational to either $\mathbb{P}^2$ or a ruled surface. The Minimal Model Program (MMP) aims at generalizing this classification in higher dimension. The problem is more complicated since there exist threefolds which do not have a smooth minimal model. For this reason we have to allow singularities and work with varieties. Given a complex projective manifold X of non negative Kodaira dimension the MMP predicts that there exists a (possibly singular) variety Y birational to X such that the canonical bundle of Y is nef. Such a variety is called a minimal model. The general strategy is as follows: one starts with a smooth complex projective manifold X. If the canonical divisor K_X is not nef then one tries to contract a negative curve on X to obtain a new variety Y which is possibly singular in general. If the singularity is mild (terminal) then one repeats the procedure on Y. If the singularity is worse then a codimension-two surgery, called a flip, is needed and one restarts the procedure finitely many times until reaching a minimal model. A challenging problem is to prove the existence of flips. This program was achieved in dimension three mainly by S. Mori [**Mor88**].

In a recent celebrated work, Birkar-Cascini-Hacon-Mckernan [**BCHM10**] showed the existence of minimal models for a large class of varieties called varieties of general type. Completing this program and extending it to Kähler varieties has attracted the attention of many research communities (algebraic, differential, analytic geometers).

J. Song and G. Tian [**ST17**] have proposed an ambitious program, combining the Minimal Model Program and Hamilton-Perelman approach to the Poincaré conjecture. The expectation is that, starting from a smooth projective variety with non-negative Kodaira dimension the flow will deform it several times, restart on the new varieties, and eventually reach a minimal model.

It is classical that the (twisted) Kähler-Ricci flow can be reduced to a nonlinear parabolic scalar equation in the potential φ, of the form

$$(\omega_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t + F(t,x,\varphi)} g dV, \tag{3.1}$$

where F is a smooth function, $(\omega_t)_{t\geq 0}$ is a smooth family of closed smooth $(1,1)$-forms, and g is a smooth positive density.

For smooth initial potential φ_0, the existence and uniqueness of the smooth flow in a maximal interval $[0, T_{\max})$ was known by [**Cao85**], [**Tsu88**], [**TZ06**]. Here $T_{\max}$ can be computed in terms of cohomology classes:

$$T_{\max} := \sup\{t \geq 0 \ : \ \{\omega_t\} \text{ is Kähler}\}.$$

We focus here on the case when X has non-negative Kodaira dimension. As proposed by Song-Tian, ideally one would like to proceed as follows :

Step 1. Show that (X, ω_t) converges to a midly singular Kähler variety (X_1, S_1) equipped with a singular *Kähler current* S_1, as $t \to T_{1,\max}$;
Step 2. Restart the flow on X_1 with initial data S_1;
Step 3. Repeat finitely many times to reach a minimal model X_r (K_{X_r} is *nef*);
Step 4. Study the long term behavior of the flow and show that (X_r, ω_t) converges to a canonical model (X_{can}, ω_{can}), as $t \to +\infty$.

2. Regularizing properties

The Song-Tian program is more or less complete in dimension ≤ 2 [**SW13**]. We next focus on step 2, running the Kähler-Ricci flow from singular data. It can also be considered as an alternative way to regularize currents. We assume F smooth and $0 < g$ is smooth but we allow the initial potential φ_0 to be singular. In [**ST17**], Song-Tian succeeded in starting the flow from continuous φ_0. Guedj-Zeriahi [**GZ17**] proved that one can start the flow from any closed positive $(1,1)$-current with zero Lelong numbers and instantly smooth it out. We have proved in [**DNL17b**] that the weak solution obtained in [**GZ17**] is unique and it is possible to start the flow from currents with positive Lelong numbers.

THEOREM 3.1. *Assume F is smooth, $0 < g$ is smooth, $\omega_t = \omega + t\chi$ is an affine family of Kähler forms for $t \in [0, T_{\max})$, and φ_0 is a ω-psh function. Then there exists a solution φ_t to the twisted Kähler-Ricci flow* (3.1) *in a Zariski open set of X. Moreover, if φ_0 has zero Lelong numbers everywhere, then the flow admits a unique smooth solution on X.*

The Zariski open subset in the above Theorem is the complement of the Lelong superlevel set $\{x \in X \ : \ \nu(\varphi_0, x) \geq c\}$, where $c > 0$ depends on $T_{\max}$ and the singularities of φ_0. By Siu's theorem these superlevel sets are analytic subsets of X. We also showed in [**DNL17b**] that the presence of positive Lelong number is an obstruction to regularizing property of the flow: if $\nu(\varphi_0, x) > 0$ for some $x \in X$ then φ_t is singular at x for small t. The proof of the first part of Theorem 3.1 uses Demailly's equisingular approximation together with a crucial relative L^∞-estimate via the generalized Monge-Ampère capacities [**DNL17a, DNL15**]. To prove the uniqueness in case the Lelong numbers are all zero we used the semigroup property of the flow, regularizing itself.

It is very natural to expect that the flow produces currents with analytic singularities.

QUESTION 9. *Assume φ_0 is a ω-psh function and let φ_t the the weak solution of the complex Monge-Ampère flow. Does φ_t have analytic singularities?*

3. Singularities encountered along the flow

The varieties appearing in the MMP are often singular but the singularities are mild. For instance, to make sense of the nefness condition of the canonical line bundle K_Y of a variety Y, one needs to define the intersection numbers of K_Y with curves on Y. This requires K_Y to be $\mathbb{Q}$-Cartier. In order to use advanced tools in differential geometry it is more convenient to work on a desingularization $\pi : X \to Y$ provided by Hironaka's work [**Hir64**]. The canonical line bundle of X is then related to that of Y by the following equivalence relation of $\mathbb{Q}$-Cartier divisors

$$K_X \equiv \pi^* K_Y + \sum a_E E,$$

where the sum runs over the exceptional divisors of π. For any locally defined multivalued canonical form η on Y the holomorphic multivalued canonical form $\pi^*\eta$ on X has poles or zeros of order a_E along E, so that the corresponding volume form decomposes as

$$\pi^*(c_n \eta \wedge \overline{\eta}) = e^{w^+ - w^-} dV(x) =: g(x) dV(x),$$

where $w^+ = \sum_{a_E > 0} a_E \log |s_E|_{h_E}$ and $w^- = \sum_{0 > a_E \geq -1} a_E \log |s_E|_{h_E}$ are quasi-plurisubharmonic with e^{w^+} continuous. The integrability of e^{-w^-} depends on the size of the coefficients a_E which in turn are encoded in the singularities of Y:

- Y has canonical singularities $\iff a_E \geq 0$, for all $E \iff g$ is continuous;
- Y has Kawamata log terminal singularities $\iff a_E > -1$, for all $E \iff g \in L^p(X)$, for some $p > 1$;
- Y has semi-log canonical singularities $\iff a_E \geq -1$.

Working on the resolution X the (weak) Kähler-Ricci flow can be written as a parabolic complex Monge-Ampère equation of the following type:

$$\text{(CMAF)} \qquad (\omega_t + dd^c u_t)^n = e^{\dot{u}_t + F(t,x,u)} g dV, \quad (t,x) \in (0,T) \times X,$$

where $T > 0$, F is a continuous increasing function in the third variable, $(\omega_t)_{t \geq 0}$ is a smooth family of smooth closed real $(1,1)$-forms, and $0 \leq g$ is a density on X.

A parabolic viscosity approach has been developed in [**EGZ16**] allowing one to study the behavior of the Kähler-Ricci flow on minimal models with positive Kodaira dimension and canonical singularities [**EGZ18**]. Finer tools are required in order to allow worst singularities.

3.1. Kawamata log-terminal singularities. In [**GLZ20a, GLZ20b**] we have developed the first steps of a parabolic pluripotential theory to study weak solutions to (CMAF) for degenerate data. We interpret the above parabolic equation on X as a second order PDE on the $(2n+1)$-dimensional manifold $X_T := (0,T) \times X$:

- the LHS becomes a positive Radon measure $(\omega_t + dd^c \varphi_t)^n \wedge dt$, which is well defined for paths $t \mapsto \varphi_t$ of bounded ω_t-psh functions [**BT76, BT82**],
- the RHS $e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x) dV(x) \wedge dt$ is a well-defined Radon measure if $t \mapsto \varphi_t(x)$ is (locally) uniformly Lipschitz.

The local side of the theory was treated in [**GLZ20a**] by a direct method using Perron upper envelopes of subsolutions, while the global side was studied in [**GLZ20b**] via an approximation scheme and a priori estimates. The theory is flexible enough to allow the forms ω_t to be merely semipositive and big with a uniform lower bound $\omega_t \geq \theta$, for a fixed semipositive and big form θ.

Theorem 3.2. *Let φ_0 be a bounded ω_0-psh function. Assume $g \in L^p(X, dV)$ for some $p > 1$ and $g > 0$ a.e. on X. There exists a unique parabolic potential $\varphi \in \mathcal{P}(X_T, \omega)$ with the following properties:*

- *$(t,x) \mapsto \varphi(t,x)$ is locally bounded in $[0,T[\times X$;*
- *$(t,x) \mapsto \varphi(t,x)$ is continuous in $]0,T[\times \mathrm{Amp}(\theta)$;*
- *$t \mapsto \varphi_t$ is locally uniformly semi-concave in $]0,T[\times X$;*
- *φ is a pluripotential solution to* (CMAF)*;*
- *$\varphi_t \to \varphi_0$ as $t \to 0^+$ in $L^1(X)$ and pointwise.*

Here $\mathrm{Amp}(\theta)$ denotes the ample locus of θ, i.e. the largest Zariski open subset of X where the cohomology class of θ behaves like a Kähler class.

It turns out that $t \mapsto \varphi_t(x) - n(t\log t - t) + Ct$ is increasing for some fixed $C > 0$. The convergence at time zero is therefore rather strong (it is e.g. uniform in case φ_0 is continuous).

The uniqueness is quite subtle, requiring several tools from the elliptic pluripotential theory developed in our previous works [**GLZ19b**], [**GLZ18**]. In case g is continuous and positive, as mentioned above there is a parabolic viscosity theory in [**EGZ16**]. In [**GLZ21**] we show that the pluripotential solution obtained in [**GLZ20b, GLZ20a**] are also viscosity solutions.

The present pluripotential approach allows us to deal with non continuous data. In [**GLZ20b**] we can, in particular, define a good notion of weak Kähler-Ricci flow on varieties with terminal singularities:

THEOREM 3.3. *Let (Y, ω_0) be a compact n-dimensional Kähler variety with log terminal singularities and trivial first Chern class ($\mathbb{Q}$-Calabi-Yau variety).*

Fix S_0 a positive closed current with bounded potentials, whose cohomology class is Kähler. The Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

exists for all times $t > 0$, and deforms S_0 towards the unique Ricci flat Kähler-Einstein current ω_{KE} cohomologous to S_0, as $t \to +\infty$.

This extends previous results of [**Cao85, Tsu88, TZ06**], avoiding any projectivity assumption on X [**ST17**], nor any restriction on the type of singularities [**EGZ16, EGZ18**].

3.2. More general singularities. In relation with the MMP it is desirable to extend [**GLZ20b**] to the case when g is in L^1 but not in L^p, $p > 1$. The standard approach via Kołodziej's technique breaks down in this case, and this makes the estimates along the flow quite delicate. In fact (quasi) plurisubharmonic solutions to complex Monge-Ampère equations with L^1-densities may be unbounded and it is reasonable to establish a relative L^∞ estimate similar to the one appeared in [**DDNL21a**], [**DNL17a**].

The non-integrable case, i.e. when $g \notin L^1(X)$, which corresponds to the log-canonical singularities considered in [**BG14**], is more challenging. To build a parabolic pluripotential theory in this context, one has to deal with unbounded quasi plurisubharmonic functions which are outside the full mass classes considered in [**BEGZ10, GZ07**]. On the elliptic side, there have been recent developments on the relative pluripotential theory [**WN19b**],[**DDNL18b, DDNL21a, DDNL21b**]. In fact, a powerful tool in pluripotential theory is the comparison principle of Bedford and Taylor [**BT76, BT82**]. Its global version [**GZ07**] is a simple consequence of the invariance of the Monge-Ampère mass. For bounded functions the later simply follows from the Stokes formula, but for unbounded ones the problem is quite involved as it requires a more sophisticated construction [**WN19b**]. Building on the monotonicity of mass, several pluripotential tools (domination principle, integration by parts, resolution of complex Monge-Ampère equations with prescribed singularities) have been established recently, see [**DDNL18b, DDNL21a, DDNL21b**], [**Xia19**], [**Tru19, Tru20**] and the previous chapter.

Given these materials we expect to prove the following:

PROBLEM 1. *Assume that X is a compact Kähler variety with semi-log canonical singularities and big canonical divisor. Then the (normalized) pluripotential Kähler-Ricci flow exists for all time and deforms any closed positive current towards the unique singular Kähler-Einstein metric.*

Parabolic Comparison principle. The elliptic comparison principle of Bedford and Taylor [**BT76, BT82**] plays a vital role in pluripotential theory. It is desirable to have its parabolic counterpart. A version of this was obtained in [**GLZ20a, GLZ20b**], allowing to prove uniqueness/stability of solutions with certain regularity under several natural assumptions. The proof of this comparison principle relies on a regularization process which works only for subsolutions and does not work for supersolutions. We believe that in order to make progress in parabolic pluripotential theory one needs to greatly improve the parabolic comparison principle. One possible direction is to regularize supersolutions by taking inf-convolution and borrowing ideas from the viscosity theory [**EGZ11, EGZ18**].

4. Big cohomology classes

We next discuss possible extensions of [**GLZ20b**] to the context of big cohomology classes, i.e. when the forms ω_t, $t > 0$ are big and not necessarily semi-positive. The latter case appears naturally in the MMP as volume non-collapsing limit of Kähler classes. To give an idea let us consider the normalized Kähler-Ricci flow on a projective manifold X of general type, i.e. such that K_X is big:

$$\frac{d}{dt}\omega_t = -\mathrm{Ric}(\omega_t) - \omega_t.$$

The flow can be run smoothly from any Kähler metric ω_0 and remains smooth in the interval $[0, T)$, where T is defined to be the maximal t for which the class $e^{-t}\{\omega_0\} + (1 - e^{-t})c_1(K_X)$ is Kähler. The flow will certainly develop singularities at T, but as the volume is non collapsing one expects that the flow will still survive after T and becomes singular. This expectation was formulated as an open question in [**FIK03**] and a more precise conjectural picture was given in [**BT12**]. One hopes to construct weak solutions for all time but uniqueness should be understood in a relative sense, i.e. in each *class of singularity types* there is a unique weak solution. The existence of a viscosity solution has been very recently proved in [**Tô17**]. As mentioned above viscosity techniques have a limited scope of applications within continuous densities. Another motivation for working with big cohomology class is the following intuition. Assume given a singular variety Y very close to a minimal model in the sense of birational geometry. To continue the road towards a minimal model, differential geometers may think of taking a resolution of singularities $\pi : X \to Y$ and working on the smooth manifold X. The difficulty here is that the singularities of X and Y are very different. But the Kähler-Ricci flow will help to fill in the gap, by deforming X towards a variety simpler than Y. The issue here is that unless K_X is nef (in which case X is the minimal model we are looking for) the smooth Kähler-Ricci flow on X stops at some finite singularity time. It is however very reasonable to hope that the evolving cohomology class remains big and our pluripotential flow exists and eventually converges to a canonical (singular) object. An illuminating example is when X is a $\mathbb{Q}$-factorial threefold with terminal singularities and pseudoeffective canonical bundle. The recent remarkable achievement of A. Höring and T. Peternell [**HP16**] shows that X admits a minimal model. In this case it is interesting to see how the pluripotential Kähler-Ricci flow (living in big cohomology classes) behaves near infinity.

A global (elliptic) pluripotential theory in big cohomology classes has been developed in the recent years in [**BEGZ10, BBGZ13, DDNL18c, DDNL18b, DDNL18a, DDNL21a, DDNL21b**]. Unlike the case of semipositive classes treated in [**GLZ20b**], one can not hope to approximate (ω_t), where for any t, ω_t is a form representing a big cohomology class, by families of Kähler forms to perturb the equation. Therefore, a possible strategy is to use the Perron method, considering the upper envelope of all pluripotential subsolutions to the Cauchy problem. The local model of this problem has been recently studied in [**GLZ20a**]. Our idea is then to combine [**GLZ20a**] and [**GLZ20b**] via a balayage process. The subsolutions in big cohomology classes are typically very singular (these may lie outside the full mass classes) and again the relative pluripotential theory developed in [**DDNL18b, DDNL21a, DDNL21b**] will certainly play an important role.

PROBLEM 2. *Assume that $(\omega_t)_{t\in[0,T)}$ is a smooth family of closed smooth real $(1,1)$-forms on X whose cohomology classes are big. Then, for each initial potential φ_0, there exists a weak* *solution* $(\varphi_t)_{t\in[0,T)}$ *to* (CMAF) *which is continuous in some Zariski open subset of* X*. Moreover, the solution is unique in each singularity class.*

Partial results in this direction have been very recently obtained by Q.T. Dang [**Dan21**] (a Ph.D student currently supervised by V. Guedj and the author).

5. On compact Hermtian manifolds

The interest towards Hermitian geometry has grown rapidly in the last decade. It is expected that Hermitian analogues of the Kähler-Ricci flow will play an important role in understanding the geometry of compact complex manifolds. Recall that any compact complex manifold admits a Hermitian metric but there are many of them which are not Kähler. It is desirable then to extend the previous works [**GLZ20b, GLZ20a**] to this more general setting.

A global pluripotential theory on compact Hermitian manifolds has been recently developed by Kołodziej, S. Dinew, N.C. Nguyen [**KN19**], [**Din16**]. A fundamental tool which is missing in this theory is the following version of the comparison principle:

$$\int_{\{u<v\}} \mathrm{MA}(v) \leq \int_{\{u<v\}} \mathrm{MA}(u),$$

which holds true on compact Kähler manifolds thanks to Stokes theorem. Interestingly, the validity of this comparison principle on a compact Hermitian manifold (X,ω) is equivalent to a geometric condition [**Chi16b**].

Due to a lack of an efficient comparison principle many estimates from the Kähler case break down. In particular, uniqueness (and stability) of solutions with $L^p, p>1$, densities was obtained in [**KN19**], [**LPT20**] under a restrictive assumption that the density is strictly positive, while the general case is widely open.

In developing an Hermitian version of the Kähler-Ricci flow with degenerate data (e.g. g is merely in $L^p, p>1$) we hope to improve the stability result in [**KN19**] [**LPT20**], and to get a better understanding of solutions to elliptic Monge-Ampère equations. A first step in this direction is to establish the existence of weak solutions to the complex Monge-Ampère flow (CMAF) on a compact Hermitian manifold (X,ω). A possible strategy to do this is to perturb the equation and establish a priori estimates as in [**GLZ20b**]. The existence of the unique smooth flow has been proved by M. Gill [**Gil11**], see also [**TW15**]. It is important to note that an adaptation of [**GLZ20b**] to the Hermitian case is quite delicate as many pluripotential tools used there are missing. On the positive side, it seems that the uniqueness of weak solutions to (CMAF) holds in the Hermitian case. This gives hope in studying the stability problem for the elliptic Monge-Ampère equation.

Problem 3. *Assume that* φ_0 *is a bounded* ω_0*-psh function on* X*. Prove that* (CMAF) *admits a continuous pluripotential solution which is locally uniformly Lipschitz in* t*. This is the unique solution with such regularity.*

It has been proved in [**Tô17**], inspired by [**GZ17**], that the complex Monge-Ampère flow instantly smoothes out any positive current with bounded potential. This interesting phenomenon has been used in the Kähler case to regularize geodesic rays along with converging radial K-energy [**DL20**] which makes progress in the Yau-Tian-Donaldson conjecture. We propose to use and generalize [**Tô17**] to make progress in pluripotential theory on Hermitian manifolds. A special metric which exists on any compact Hermitian manifold is the Gauduchon metric [**Gau77**]. This metric was proved to be very useful in solving (elliptic) complex Monge-Ampère equations. An interesting question is whether there is a notion of Gauduchon currents. More specifically, given a positive current $\omega+dd^c\varphi$ with bounded potential, is there a function G such that $e^G(\omega+dd^c\varphi)^{n-1}$ is dd^c-closed? A possible strategy is to look at the Monge-Ampère flow ω_t starting from T which smoothes it out immediately. Then, for each $t > 0$ there is a Gauduchon metric in the conformal class of ω_t. The goal is then to estimate G_t along the flow. Another related problem that we have in mind is the uniqueness of the Monge-Ampère constants/measures. As shown in [**TW10**], [**KN16**], [**KN19**], given a density $0 < f \in L^p$, $p > 1$ with positive mass, one can solve the complex Monge-Ampère equation

$$(\omega + dd^c u)^n = c_f f \omega^n,$$

where $c_f > 0$ is a constant. In contrast with the Kähler case, here the constant c_f depends heavily on f. A stability result for this constant has been recently established in [**LPT20**]. Using the regularizing property of the Monge-Ampère flow we expect to give an answer to following question.

Question 10. *Assume* $u, v \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ *and* $(\omega + dd^c u)^n \leq (\omega + dd^c v)^n$. *Can we infer* $(\omega + dd^c u)^n = (\omega + dd^c v)^n$*?*

The attempts to establish stability of the complex Monge-Ampère equations in [**KN19**], [**LPT20**] somehow boil down to answering the above question. In the Kähler case the answer is yes since the positive measures $(\omega + dd^c u)^n$ and $(\omega + dd^c v)^n$ have the same total mass. If we have a positive answer to Question 10 then we expect to solve the following

Problem 4. *Assume that* $0 \leq f, g \in L^p(X)$ *for some* $p > 1$. *Assume* u *and* v *are bounded* ω*-psh functions such that*

$$(\omega + dd^c u)^n = f\omega^n \ , \ \ (\omega + dd^c v)^n = g\omega^n, \ \sup_X u = \sup_X v = 0.$$

Prove that $|u - v| \leq C\|f - g\|_p^{1/n}$, *for a uniform constant* $C > 0$.

CHAPTER 4

Geodesic rays and constant scalar curvature Kähler metrics

This chapter is concerned with the study of geodesic rays and constant scalar curvature metrics. We have obtained several results related to this topic in [**BDL17, BDL20**], [**DL20**], [**DLR20**], [**DDNL21b**], [**DNGL21**].

1. Plurisubharmonic geodesics

Let X be a compact connected Kähler manifold of dimension n and fix a Kähler form ω. By the dd^c-lemma, any Kähler metric cohomologous to ω can be written as $\omega_u := \omega + dd^c u$, where u is a smooth real function. We let

$$\mathcal{H} := \{\varphi \in C^\infty(X, \mathbb{R}) \ : \ \omega + dd^c\varphi > 0\}$$

denote the space of Kähler potentials of ω. This is an open convex subset of the Fréchet space $C^\infty(X, \mathbb{R})$, so it is also a Fréchet manifold. The tangent space $\mathcal{H}_\varphi$ can be identified with $C^\infty(X, \mathbb{R})$. Mabuchi [**Mab87**] introduced a Riemannian structure on $\mathcal{H}$ by defining

$$\langle u, v\rangle_\varphi := \int_X uv(\omega + dd^c\varphi)^n.$$

A geodesic is a smooth path $(\varphi_t)_{t\in[0,1]}$ minimizing the energy

$$\int_0^1 \int_X |\dot\varphi_t|^2(\omega + dd^c\varphi_t)^n dt.$$

Semmes [**Sem92**] and Donaldson [**Don99**] have independently discovered an important interpretation of the geodesic equation as a homogeneous degenerate complex Monge-Ampère equation

$$(\pi^*\omega + dd^c\Phi)^{n+1} = 0, \text{ on } X \times D,$$

where $D = \{z \in \mathbb{C},\ 1 < |z| < e\}$, $\Phi(x, z) := \varphi_{\log|z|}(x)$, and $\pi : X\times D \to X$ is the projection on the first factor. This equation can be understood in the weak sense of measures following pluripotential theory, and by Berndtsson [**Ber15a**] there exists a unique weak solution $(\varphi_t)_{t\in[0,1]}$ connecting bounded ω-psh functions φ_0, φ_1. Even if the two end points φ_0, φ_1 are in $\mathcal{H}$, the solution Φ is not necessarily in $\mathcal{H}$. Chen [**Che00**] proved that Φ is $C^{1,\bar{1}}$ on $X \times D$. Counter examples of Darvas-Lempert [**DL12**], Lempert-Vivas [**LV13**] show that the $C^{1,1}$ regularity recently obtained by Chu-Tosatti-Weinkove [**CTW18**] is optimal. In the same work Chen [**Che00**] proved that $\mathcal{H}$ equipped with the Mabuchi d_2-distance

$$d_2(\varphi_0, \varphi_1) := \inf\left\{\int_0^1 \left(\int_X |\dot\varphi_t|^2(\omega + dd^c\varphi_t)^n\right)^{1/2} dt\right\},$$

where the infimun is taken over all smooth paths (φ_t) connecting φ_0 to φ_1, is a metric space, which is non-positively curved in the sense of Alexandrov, and $d_2(\varphi_0, \varphi_1)$ is realized by the weak geodesic segment Φ:

$$d_2(\varphi_0, \varphi_1)^2 = \int_X |\dot\varphi_t|^2(\omega + dd^c\varphi_t)^n,\ \forall t \in [0, 1].$$

Darvas [**Dar17a**] introduced L^p-type Finsler metrics d_p, $p \geq 1$, on $\mathcal{H}$:

$$d_p(\varphi_0, \varphi_1) := \inf \left\{ \int_0^1 \left(\int_X |\dot{\varphi}_t|^p (\omega + dd^c \varphi_t)^n \right)^{1/p} dt \right\}.$$

Moreover, given φ_0 and φ_1 in $\mathcal{H}^{1,\bar{1}}$, the set of potentials in $\mathrm{PSH}(X, \omega)$ whose Laplacian is bounded, according to W. He [**He15**] the geodesic segment is also in $\mathcal{H}^{1,\bar{1}}$, and by Darvas [**Dar15**]

$$d_p(\varphi_0, \varphi_1)^p := \int_X |\dot{\varphi}_t|^p (\omega + dd^c \varphi_t)^n, \ t \in [0,1]. \tag{4.1}$$

As shown by an example in [**Dar15**], one can not expect this formula to hold for arbitrary end points φ_0, φ_1. It is thus natural to ask

QUESTION 11. *Under what condition on φ_0, φ_1, does the formula* (4.1) *hold? If φ_0, φ_1 are Hölder continuous, is φ_t also Hölder continuous?*

QUESTION 12. *If $\omega_{\varphi_0}^n = f_0 dV$ and $\omega_{\varphi_1}^n = f_1 dV$ have L^1-densities, does $\omega_{\varphi_t}^n$ have L^1-density?*

The space $(\mathcal{H}, d_p)$ thus obtained are geodesic metric spaces but they are not complete. Recall that a geodesic metric space (E, d) is a metric space for which any two points can be connected with a geodesic. By a geodesic connecting two points $a, b \in E$ we understand a curve $\alpha : [0,1] \to E$ such that $\alpha(0) = a$, $\alpha(1) = b$ and

$$d(\alpha(t_1), \alpha(t_2)) = |t_1 - t_2| d(a, b),$$

for any $t_1, t_2 \in [0,1]$. Understanding the completion of these spaces turns out to be an important problem in Kähler geometry. Guedj [**Gue14**] conjectured that the completion of $(\mathcal{H}, d_2)$ is the finite energy space $\mathcal{E}^2(X, \omega)$ studied earlier by Guedj-Zeriahi [**GZ07**]. In his landmark work [**Dar17a**], [**Dar15**], Darvas confirmed this conjecture and extended it for the d_p distance. A novel ingredient in Darvas' approach is the following rooftop envelope:

$$P_\omega(u, v) = P_\omega(\min(u, v))$$

for $u, v \in \mathrm{PSH}(X, \omega)$ (see also [**DR16**]). An ω-psh function u belongs to $\mathcal{E}^p$ if $u \in \mathcal{E}(X, \omega)$ and $\int_X |u|^p (\omega + dd^c u)^n < +\infty$. For each $u \in \mathcal{E}^p$, by Demailly [**Dem92**], Błocki-Kołodziej [**BK07**], we can find a decreasing sequence (u_j) in $\mathcal{H}$ such that $u_j \searrow u$. If $u_0, u_1 \in \mathcal{E}^p$, the sequence of geodesic segments $(u_{j,t})_{t \in [0,1]}$ is decreasing in j as follows from the comparison principle. As the function $P(u_0, u_1) \in \mathcal{E}^p$ stays below these geodesic segments, the limiting geodesic (u_t) belongs to $\mathcal{E}^p$. One can then show that (u_t) does not depend on the choice of the approximants, and it is called the *psh geodesic* connecting u_0 to u_1. By Darvas [**Dar15, Dar17a**], the limit

$$d_p(u_0, u_1) := \lim_{j \to +\infty} d_p(u_{j,0}, u_{j,1})$$

exists and it does not depend on the choice of the sequences $(u_{j,0}), (u_{j,1})$. Moreover, $(\mathcal{E}^p, d_p)$ is a geodesic metric space which is complete and it is the completion of $(\mathcal{H}, d_p)$. Darvas' construction has been generalized to big and nef cohomology classes in [**DNG18**], [**DNL20**].

Another important achievement of [**Dar15**] is that d_p is uniformly comparable to certain pluripotential energy:

$$C(n)^{-1} d_p(u, v)^p \leq \int_X |u - v|^p (\omega_u^n + \omega_v^n) \leq C(n) d_p(u, v)^p.$$

2. Uniform convexity and uniqueness of geodesic segments

For any $p \in [1,\infty)$ it was shown in [**CC21b**, Theorem 1.5] that the metrics d_p are "convex": if $[0,1] \ni t \to u_t, v_t \in \mathcal{E}^p$ are two psh geodesic segments then

$$d_p(u_\lambda, v_\lambda) \le (1-\lambda)d_p(u_0,v_0) + \lambda d_p(u_1,v_1),\ \lambda \in [0,1]. \tag{4.2}$$

This property is called Buseman convexity in the metric geometry literature [**Jos97**, Section 2.2], going back to [**Bus55**]. In the particular case $p=1$, (4.2) was established in [**BDL17**, Proposition 5.1], having applications to the convergence of the weak Calabi flow. In case $p=2$, (4.2) follows from the fact that $(\mathcal{E}^2, d_2)$ is a complete CAT(0) metric space, as shown in [**Dar17a**, Theorem 1], building on estimates of [**CC02**, Theorem 1.1].

The CAT(0) property consists of the following estimate: if $u \in \mathcal{E}^2$ and $[0,1] \ni t \to v_t \in \mathcal{E}^2$ is a psh geodesic segment then, for all $\lambda \in [0,1]$,

$$d_2(u,v_\lambda)^2 \le (1-\lambda)d_2(u,v_0)^2 + \lambda d_2(u,v_1)^2 - \lambda(1-\lambda)d_2(v_0,v_1)^2. \tag{4.3}$$

As is well-known, (4.3) implies (4.2) [**Jos97**, Prop 2.3.2]. When restricting to a toric Kähler manifold and toric Kähler metrics, the spaces $(\mathcal{E}^p, d_p)$ are isometric to the flat L^p metric spaces of convex functions defined on a convex polytope of $\mathbb{R}^n$ [**DNG18**, Section 6]. It is well known however that CAT(0) Banach spaces are in fact Hilbert spaces [**BH99**], evidencing that only $(\mathcal{E}^2, d_2)$ can be CAT(0).

Despite this, in [**DL20**] we have shown that adequate generalizations of the CAT(0) inequality (4.3) do hold for the d_p metrics, in case $p>1$. These can be viewed as the Kähler analogs of classical inequalities of Clarkson and Ball–Carlen–Lieb, regarding the uniform convexity of L^p spaces [**Cla36, BCL94**]. Consequently, the metric spaces $(\mathcal{E}^p, d_p)$ are *uniformly convex* for $p>1$:

THEOREM 4.1. *Let $p \in (1,\infty)$. Suppose that $u \in \mathcal{E}^p$, $\lambda \in [0,1]$ and $[0,1] \ni t \to v_t \in \mathcal{E}^p$ is a psh geodesic segment. Then the following hold:*
(i) $d_p(u,v_\lambda)^2 \le (1-\lambda)d_p(u,v_0)^2 + \lambda d_p(u,v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0,v_1)^2$, if $1 < p \le 2$.
(ii) $d_p(u,v_\lambda)^p \le (1-\lambda)d_p(u,v_0)^p + \lambda d_p(u,v_1)^p - \lambda^{\frac{p}{2}}(1-\lambda)^{\frac{p}{2}}d_p(v_0,v_1)^p$, if $2 \le p$.

In the particular case $p=2$ this result recovers the inequalities of Calabi–Chen [**CC02**], however our proof of Theorem 4.1 is very different from the argument in [**CC02**], as the differentiation of d_p metrics is problematic for $p \ne 2$.

As shown by Darvas [**Dar15**] d_1-geodesic segments connecting the different points of $(\mathcal{E}^1_\omega, d_1)$ are not unique. However, as a consequence of the above result it follows that uniqueness of d_p-geodesic segments does hold in case $p>1$.

3. Quantization of $\mathcal{E}^p$

In this section we assume that $\{\omega\}$ is integral, i.e. ω is the curvature of an ample holomorphic line bundle L over X. A major theme in Kähler geometry, going back to a problem of Yau [**Yau87**, p. 139] and work of Tian [**Tia90**] thirty years ago, has been the approximation (or "quantization") of the infinite-dimensional space of Kähler potentials $\mathcal{H}$ by the finite-dimensional spaces

$$\mathcal{H}_k := \{\text{positive Hermitian forms on } H^0(X,L^k)\},$$

since the $\mathcal{H}_k$ can be identified as *subspaces* of $\mathcal{H}$ consisting of (algebraic) Fubini–Study metrics. It was suggested by Donaldson that the geometry of $\mathcal{H}$ should be approximated by the geometry of $\mathcal{H}_k$ [**Don01**, p. 483]. In [**DLR20**] we investigate the quantization problem for $\mathcal{H}$ equipped with Finsler metrics d_p, for $p \ge 1$.

We first consider different L^p Finsler structures on the space $\mathcal{P}_n$ of positive Hermitian n-by-n matrices. For any $h \in \mathcal{P}_n$, the tangent space is the space of all Hermitian n-by-n matrices. There is a classical Riemannian metric on $\mathcal{P}_n$ given by

$$\langle \eta, \nu \rangle|_h := \frac{1}{n}\mathrm{Tr}\left[h^{-1}\eta h^{-1}\nu\right], \eta, \nu \in T_h\mathcal{P}_n,$$

and, by a standard variational argument [**Kob14**, p. 195], geodesics with endpoints $h_0, h_1 \in \mathcal{P}_n$ are solutions of

$$\frac{d}{dt}\left(h_t^{-1} \cdot \dot{h}_t\right) = 0, \quad t \in [0,1], \tag{4.4}$$

thus

$$d_{2,\mathcal{P}_n}(h_0, h_1) = \left[\frac{1}{n}\sum_{j=1}^{n} |\lambda_j|^2\right]^{\frac{1}{2}},$$

where

$$e^{\lambda_1}, \dots, e^{\lambda_n}$$

are the eigenvalues of $h_0^{-1}h_1$. For any $p \geq 1$ we introduce Finsler structures on $\mathcal{P}_n$,

$$||\nu||_{p,h} := \left[\frac{1}{n}\mathrm{Tr}\left(|h^{-1}\nu|^p\right)\right]^{\frac{1}{p}}, \ \eta, \nu \in T_h\mathcal{P}_n.$$

We denote by $d_{p,\mathcal{P}_n}$ the resulting path length metric on $\mathcal{P}_n$.

THEOREM 4.2. *Let $p \geq 1$. Solutions of (4.4) are metric geodesics of $(\mathcal{P}_n, d_{p,\mathcal{P}_n})$, thus*

$$d_{p,\mathcal{P}_n}(h_0, h_1) = \left[\frac{1}{n}\sum_{j=1}^{n} |\lambda_j|^p\right]^{\frac{1}{p}}, \quad h_0, h_1 \in \mathcal{P}_n,$$

and therefore $(\mathcal{P}_n, d_{p,\mathcal{P}_n})$ is a geodesic metric space.

Note that the geodesic equation is therefore *independent of p*. This result parallels an analogous result in the infinite-dimensional setting of $\mathcal{H}$: the L^p Finsler structures of $\mathcal{H}$ have common geodesics as well [**Dar15**, Theorem 1].

Next, we fix h_L, a Hermitian metric on L whose curvature is $\omega = \Theta(h_L) = -dd^c \log h_L > 0$. We denote by h_L^k the k-th tensor product of the metric on L^k, the k-th tensor product of L. Define the Hilbert map $\mathrm{H}_k : \mathcal{E}^p \to \mathcal{H}_k$ by

$$\mathrm{H}_k(u)(s,s) := \int_X h_L^k(s,s)e^{-ku}\omega^n.$$

This is not quite the well-known Hilbert map often denoted by $Hilb_k$ in the literature [**Don05**], since we integrate against ω^n instead of ω_u^n. In fact, for any $u \in \mathcal{E}^p \setminus \mathcal{E}^q$, $q > p$, the integral $\int_X e^{-ku}\omega_u^n$ is seen to be infinite for all k. On the other hand, since elements of $\mathcal{E}^p$ have zero Lelong numbers, the map H_k above is well-defined. In particular, in the context of the L^p metric completions $\mathcal{E}^p$, this definition of H_k is not only the most natural, but also the only one that makes sense.

In the opposite direction, the classical map $\mathrm{FS}_k : \mathcal{H}_k \to \mathcal{H} \subset \mathcal{E}^p$ sends an inner product G to the associated Fubini–Study metric restricted to X,

$$\mathrm{FS}_k(G) := \frac{1}{k}\log\sum_{j=1}^{\mathrm{d}_k} |e_j|^2_{h_L^k},$$

where $\{e_j\}_{j=1,\dots,\mathrm{d}_k}$ is a (any) G–orthonormal basis of $H^0(X, L^k)$. Equivalently, $\mathrm{FS}_k(G)$ can be thought of as a Bergman kernel for which the classical extremal characterization will prove handy:

$$\mathrm{FS}_k(G)(x) = \sup_{s \in H^0(X,L^k), G(s,s)=1} \frac{1}{k}\log|s(x)|^2_{h_L^k}.$$

Our main result in [**DLR20**] shows that the geometry of $(\mathcal{E}^p, d_p)$ can be approximated by that of finite dimensional spaces:

THEOREM 4.3. *For any $p \geq 1$ the following hold:*
(i)(Quantization of points) For $v \in \mathcal{E}^p$ we have

$$\lim_k d_p(\mathrm{FS}_k \circ \mathrm{H}_k(v), v) = 0.$$

(ii)(Quantization of distance) For $v_0, v_1 \in \mathcal{E}^p$ we have

$$\lim_k d_{p,k}(\mathrm{H}_k(v_0), \mathrm{H}_k(v_1)) = d_p(v_0, v_1).$$

(iii)(Quantization of geodesics) Suppose $u_0, u_1 \in \mathcal{E}^p$ and $[0,1] \ni t \to u_t \in \mathcal{E}^p$ is the L^p-finite-energy geodesic connecting u_0, u_1. Let $[0,1] \ni t \to U^k_t \in \mathcal{H}_k$ be the L^p-Finsler geodesic joining $U^k_0 = \mathrm{H}_k(u_0)$ and $U^k_1 = \mathrm{H}_k(u_1)$, solving (4.4). Then

$$\lim_k d_p(\mathrm{FS}_k(U^k_t), u_t) = 0 \quad \text{for any} \quad t \in [0,1].$$

Theorem 4.3 (i) is the geometric pluripotential theory analogue of the asymptotic expansion of the smooth Bergman kernel (i.e., for *smooth* v) due to the work of Boutet de Monvel–Sjostrand, Catlin, Tian, and Zelditch [**BdMS76, Cat99, Tia90, Zel98**] (for convergence to equilibrium in case of non-positive metrics see [**Ber09, DMM16**]).

Similarly, part (ii) for *smooth* v_0, v_1 and $p = 2$ is a result of Chen-Sun [**CS12**, Theorem 1.1], using a slightly different H_k map (with an alternative proof due to Berndtsson [**Ber18**, Theorem 1.1] using the language of spectral measures).

Finally, part (iii) for *smooth* u_0, u_1 and $p = 2$ is a theorem of Berndtsson [**Ber18**, Theorem 1.2], extending previous work of Phong-Sturm [**PS06**, Theorem 1]. For smooth potentials u_0, u_1, Berndtsson proved actually C^0-convergence of $FS_k(U^k_t)$, which implies d_2-convergence by [**Dar15**, Theorem 3]. Similarly, in case of toric manifolds, Song-Zelditch [**SZ10**] proved C^2-convergence of $FS_k(U^k_t)$. We emphasize though that the d_p-convergence in our result is optimal since a typical element of $\mathcal{E}^p$ is unbounded.

Compared to the above mentioned works, in the absence of smoothness, the well known asymptotic expansion of the smooth Bergman kernel will have only very limited use, and instead we will have to rely almost exclusively on pluripotential theoretic and complex-algebraic tools. In addition to techniques in finite-energy pluripotential theory, our two cornerstones are the Ohsawa–Takegoshi extension theorem [**OT87**] and the quantized maximum principle of Berndtsson [**Ber18**]. However to use these latter theorems, one needs to work with strongly positive $(1,1)$ currents. This represents a significant difficulty, as finite-energy currents do not satisfy such positivity property in general, and we need to develop a suitable approximation technique using strongly positive currents. This is achieved by using quasi-psh envelopes inspired by our ideas in [**DDNL21b**].

Given that the pluripotential side of our arguments above are more or less well developed in the context of big cohomology classes, it is natural to investigate

PROBLEM 5. *Prove similar results for $\mathcal{E}^p(X, \theta)$, where $\{\theta\} = c_1(L)$ is a big cohomology class.*

4. Geodesic rays and singularity types

A geodesic ray $\{u_t\}$ is a continuous curve $[0, +\infty) \ni t \mapsto u_t \in \mathrm{PSH}(X, \omega)$ such that the restriction of u_t on each finite interval is a psh geodesic segment. A bounded (respectively $\mathcal{E}^p$) psh geodesic ray is a geodesic ray consisting of bounded (respectively $\mathcal{E}^p$) ω-psh functions. The space of bounded (respectively $\mathcal{E}^p$) geodesic rays will be denoted by $\mathcal{R}^\infty$ (respectively $\mathcal{R}^p$). Given a bounded psh geodesic ray $\{u_t\}$ emanating from 0, we can define the Legendre transform

$$\hat{u}_\tau := \inf_{t\geq 0}(u_t - t\tau),\ \tau\in\mathbb{R}.$$

As shown in [**Dar17b**] the function $\hat{u}_\tau$ is either identically $-\infty$ or a model potential. By construction the curve $\tau\mapsto\hat{u}_\tau$ is concave and there exists a constant $C=C(u)$ such that for $\tau\leq -C$, $\hat{u}_\tau\equiv 0$, and for $\tau>C$, $\hat{u}_\tau\equiv-\infty$. Such a curve was called a (maximal) test curve in [**RWN14**]. Conversely, starting from a test curve ψ_τ, one can take the inverse Legendre transform

$$\check{\psi}_t := \sup_{\tau\in\mathbb{R}}(\psi_\tau + t\tau),\ t\geq 0.$$

There is no need to take the upper semicontinuous regularization as the function $\check{\psi}_t$ is already upper semicontinuous. It was shown in [**RWN14**] that $\check{\psi}_t$ is a bounded psh geodesic ray emanating from 0. In [**DDNL18a**], we show that this construction can be generalized for big cohomology classes. The correspondence between maximal test curves and geodesic rays was further exploited in [**DL20**] to approximate $\mathcal{E}^1$ geodesic rays with bounded ones, leading to a partial resolution of Donaldson's geodesic stability conjecture [**Don99**].

Let $\mathcal{R}^p$ denote the space of $\mathcal{E}^p$ psh geodesic rays starting from 0. We define the d_p^c distance between two rays $u=(u_t)$ and $v=(v_t)$ by setting

$$d_p^c(u,v) = \lim_{t\to+\infty}\frac{d_p(u_t,v_t)}{t}.$$

The existence of the limit is a consequence of Chen-Cheng's convexity result (4.2). One of the main results in [**DL20**] shows that

THEOREM 4.4. *$(\mathcal{R}^p, d_p^c)$ is a complete geodesic metric space for any $p\in[1,\infty)$.*

When $p>1$, the d_p^c-geodesic segments can be constructed directly using uniform convexity (Theorem 4.1). In case $p=1$, in the absence of uniform convexity, the construction of d_1^c-geodesic segments is done using an approximation procedure.

The comparison of singularity types of ω-psh functions is easily seen to yield an equivalence relation, whose equivalence classes $[w], w\in \mathrm{PSH}(X,\omega)$ give rise to the space of singularity types $\mathcal{S}(X,\omega)$. This latter space plays an important role in transcendental algebraic geometry, as its elements represent the building blocks of multiplier ideal sheaves, log-canonical thresholds, etc., bridging the gap between the algebraic and the analytic viewpoint on the subject.

As is well known $\mathrm{PSH}(X,\omega)$ has a natural complete metric space structure given by the L^1 metric. However the latter does not naturally descend to $\mathcal{S}(X,\omega)$ making the study of variation of singularity type quite awkward and cumbersome. Indeed, reviewing the literature, "convergence of singularity types" is only discussed in an ad-hoc manner, under stringent conditions on the potentials involved.

On the other hand, "approximating" an arbitrary singularity type $[u]$ with one that is much nicer goes back to the beginnings of the subject. Perhaps the most popular of these approximation procedures is the one that uses Bergman kernels, as first advocated in this context by Demailly [**Dem92**]. Here, using Ohsawa-Takegoshi type theorems one obtains a (mostly decreasing) sequence $[u_{j,B}]$ that in favorable circumstances approaches $[u]$ in the sense that multiplier ideal sheaves, log-canonical thresholds, vanishing theorems, intersection numbers etc. can be recovered in the limit. Still, no metric topology seems to be known that could quantify the effectiveness or failure of the "convergence" $[u_{j,B}]\to[u]$.

In [**DDNL21b**] we have introduced a natural (pseudo)metric $d_{1,\mathcal{S}}$ on $\mathcal{S}(X,\omega)$ and pointed out that it fits well with some already existing approaches in the literature. The precise definition of $d_{\mathcal{S}}$ uses a geodesic construction due to Darvas [**Dar17b**]: given $u\in\mathrm{PSH}(X,\omega)$ we consider subgeodesic rays u_t^s where for $0 \leq t \leq s$, u_t^s is the geodesic segment connecting 0 to $\max(u, -s)$, and for $t > s$, we set $u_t^s = \max(u, -t)$. These subgeodesic rays increase as $s \to +\infty$ to a bounded geodesic ray u_t encoding singularity types of u. We define

$$d_{1,\mathcal{S}}(u,v) := \lim_{t\to+\infty} \frac{d_1(u_t, v_t)}{t}.$$

Since the ray u_t does not depend on the potential u in the singularity class $[u]$, the pseudo-distance $d_{1,\mathcal{S}}$ descends to the space of singularity types. Given the convexity of d_p, $p \geq 1$, it is possible to define $d_{p,\mathcal{S}}(u,v)$. By the construction in [**DDNL18a**] our study carries over the context of big cohomology classes.

We have shown in [**DDNL21b**] that there is a uniform constant $C > 1$ only dependent on $\dim_{\mathbb{C}} X$ such that:

$$d_{1,\mathcal{S}}([u],[v]) \leq \sum_{j=0}^{n} \left(2\int_X \theta^j_{V_\theta} \wedge \theta^{n-j}_{\max(u,v)} - \int_X \theta^j_{V_\theta} \wedge \theta^{n-j}_v - \int_X \theta^j_{V_\theta} \wedge \theta^{n-j}_u \right) \leq C d_{1,\mathcal{S}}([u],[v]).$$

One amazing thing one can read from these inequalities is that the expression in the middle will also satisfy the quasi-triangle inequality.

It was proved in [**DDNL21b**] that $d_{1,\mathcal{S}}([u],[v]) = 0$ when the singularities of u and v are essentially indistinguishable (the Lelong numbers, multiplier ideal sheaves, mixed masses of $[u]$ and $[v]$ are the same). More precisely, $d_{1,\mathcal{S}}([u],[v]) = 0$ if and only if u and v belong to the same relative full mass class, as introduced in the previous chapter. In particular, $u \in \mathcal{E}(X,\theta)$ if and only if $d_{1,\mathcal{S}}([u],[V_\theta]) = 0$. Consequently, the degeneracy of $d_{1,\mathcal{S}}$ is quite natural!

Given the $d_{1,\mathcal{S}}$-continuity of $[u] \to \int_X \theta_u^n$ it is quite natural to introduce the following subspaces for any $\delta \geq 0$:

$$\mathcal{S}_\delta(X,\theta) := \left\{ [u] \in \mathcal{S}(X,\theta) : \int_X \theta_u^n \geq \delta \right\}.$$

These spaces are $d_{1,\mathcal{S}}$-closed, and according to our main result in [**DDNL21b**] they are also complete:

THEOREM 4.5. *For any $\delta > 0$ the space $(\mathcal{S}_\delta(X,\theta), d_{1,\mathcal{S}})$ is complete.*

Unfortunately the space $(\mathcal{S}(X,\theta), d_{1,\mathcal{S}})$ is not complete. This is quite natural however, as issues may arise if the non-pluripolar mass vanishes in the $d_{1,\mathcal{S}}$-limit. It is natural to ask

QUESTION 13. *Is the space $(\mathcal{S}(X,\theta), d_{p,\mathcal{S}})$ complete for $p > 1$?*

As alluded to above, in general L^1-convergence of potentials (or even convergence in capacity) does not imply $d_{1,\mathcal{S}}$–convergence of their singularity types. However if $u_j \nearrow u$ pointwise a.e. then $d_{1,\mathcal{S}}([u_j],[u]) \to 0$.

Suppose that $u, v \in \mathrm{PSH}(X,\theta)$ is such that $P(u,v) \in \mathrm{PSH}(X,\theta)$. Then $[\max(u,v)]$ and $[P(u,v)]$ represent the maximum and the minimum of the singularity types $[u],[v]$ respectively, and these four singularity types form a "diamond" in the semi-lattice $\mathcal{S}(X,\theta)$. The following inequality between the masses of these potentials is of independent interest:

THEOREM 4.6. *Suppose that $u, v, P(u,v) \in \mathrm{PSH}(X,\theta)$. Then*

$$\int_X \theta_u^n + \int_X \theta_v^n \leq \int_X \theta^n_{\max(u,v)} + \int_X \theta^n_{P(u,v)}.$$

In case $\dim X = 1$, the above inequality is actually an equality, however strict inequality may occur if $\dim X \geq 2$.

Applications to multiplier ideal sheaves. For $[v] \in \mathcal{S}(X,\theta)$ we denote by $\mathcal{J}[v]$ the multiplier ideal sheaf associated to the singularity type $[v]$. Recall that $\mathcal{J}[v]$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-v}$ is locally integrable on X. Providing a positive answer to the Demailly strong openness conjecture [**DK01**], Guan–Zhou have shown in [**GZ15b, GZ15a**] that for any u_j, u psh such that $u_j \nearrow u$ a.e. we have that $\mathcal{J}[u_j] = \mathcal{J}[u]$ for $j \geq j_0$, with a partial result obtained earlier by Berndtsson [**Ber15b**]. In [**DDNL21b**] we extend the scope of this theorem to the global context, providing a result that uses $d_{\mathcal{S}}$-convergence and avoids the condition $u_j \leq u$:

THEOREM 4.7. *Let $[u], [u_j] \in \mathcal{S}(X,\theta),\ j \geq 0$, such that $d_{1,\mathcal{S}}([u_j],[u]) \to 0$. Then there exists $j_0 \geq 0$ such that $\mathcal{J}[u] \subseteq \mathcal{J}[u_j]$ for all $j \geq j_0$.*

The proof of this theorem involves an application of Theorem 4.6 and the local Guan–Zhou result for increasing sequences [**GZ15b, GZ15a**]. Lastly, since $u_j \leq u$ trivially gives $\mathcal{J}[u_j] \subseteq \mathcal{J}[u]$, our theorem contains the global version of the Guan–Zhou result for increasing sequences of θ-psh potentials.

Motivated by a possible local analog of Theorem 4.7 it would be interesting to see if a local version of the $d_{1,\mathcal{S}}$ metric exists on the space of singularity types of local psh potentials.

Note that equality in the inclusion $\mathcal{J}[u] \subseteq \mathcal{J}[u_j]$ of Theorem 4.7 can not be expected in general. Indeed, $d_{1,\mathcal{S}}([\lambda u],[u]) \to 0$ as $\lambda \nearrow 1$ for any $u \leq 0$, however if u has log type singularity at some $x \in X$, but is locally bounded on $X \setminus \{x\}$, then $\mathcal{J}[u] \subsetneq \mathcal{J}[\lambda u] = \mathcal{O}_X, \lambda \in (0,1)$.

5. Mabuchi K-energy and cscK metrics

The scalar curvature of a Kähler metric ω is the trace of its Ricci form:

$$\mathrm{Scal}(\omega) := n\frac{\mathrm{Ric}(\omega)\wedge\omega^{n-1}}{\omega^n} \in C^\infty(X,\mathbb{R}).$$

A metric ω is cscK (constant scalar curvature) if the scalar curvature $\mathrm{Scal}(\omega) = \bar{S}$ is constant. A simple application of Stokes theorem ensures that this constant depends only on the first Chern class and $\{\omega\}$:

$$\bar{S} = \bar{S}_\omega = nV^{-1}\int_X c_1(X)\wedge\{\omega\}^{n-1},$$

where $V = \int_X \omega^n$ is the volume of ω. A central problem in Kähler geometry is to determine whether a class $\{\omega\}$ contains a cscK metric. When $c_1(X)$ is proportional to $\{\omega\}$, a metric $\omega_u \in \{\omega\}$ is cscK if and only if it is Kähler-Einstein, and finding a Kähler-Einstein metric boils down to studying a complex Monge-Ampère equation. In general, cscK metrics do not seem to be solutions of Monge-Ampère equations. From a variational approach initiated by [**Mab87**] and [**Don99**], looking for a cscK metric $\omega_u \in \{\omega\}$ is equivalent to minimizing the Mabuchi K-energy functional. We recall below its definition as well as several related energy functionals.

Given a positive closed (1,1)-form χ, the functional $J_\chi : \mathcal{E}^1 \to \mathbb{R}$ is defined as follows:

$$J_\chi(u) = E_\chi(u) - \frac{1}{V}\left(\int_X \chi\wedge\omega^{n-1}\right)E(u),$$

where E and E_χ are the Monge-Ampère energy and its "χ-contracted" version:

$$E(u) = \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X u\omega_u^j\wedge\omega^{n-j},\ E_\chi(u) = \frac{1}{nV}\sum_{j=0}^{n-1}\int_X u\chi\wedge\omega_u^j\wedge\omega^{n-1-j}.$$

The following bi-functional was introduced in [**BBE$^+$19**]:

$$\mathcal{I}(u,v) := \int_X (u-v)(\omega_v^n - \omega_u^n),\ u,v \in \mathcal{E}^1\cap E^{-1}(0).$$

A simple integration by parts shows that $\mathcal{I}(u,v) \geq 0$, and the domination principle ensures that $\mathcal{I}$ is non-degenerate, i.e. $\mathcal{I}(u,v) = 0$ if and only if $u = v$. An elementary calculation gives the following useful estimates:

$$\frac{1}{n(n+1)}\mathcal{I}(u,v) \leq J_{\omega_u}(v) - J_{\omega_u}(u) \leq \mathcal{I}(u,v),$$

for all $u, v \in \mathcal{E}^1_0 := \mathcal{E}^1 \cap E^{-1}(0)$. By Chen and Tian the K-energy can be expressed explicitly:

$$\mathcal{M}(u) := \bar{S}E(u) - nE_{\mathrm{Ric}(\omega)}(u) + \mathrm{Ent}(\omega^n, \omega_u^n),$$

where $\mathrm{Ent}(\nu, \mu)$ is the relative entropy of the measure μ with respect to the measure ν defined as

$$\mathrm{Ent}(\nu,\mu) = \int_X \log(\mu/\nu) d\mu,$$

if μ is absolutely continuous with respect to ν, and $+\infty$ otherwise. A simple integration by parts shows that

$$\frac{d\mathcal{M}(u_t)}{dt} = \int_X \dot{u}_t(\bar{S} - S(\omega_{u_t}))(\omega + dd^c u_t)^n$$

for any smooth path $u_t \in \mathcal{H}$. As a consequence cscK potentials of ω are critical points of $\mathcal{M}$. More generally, one can consider the twisted K-energy $\mathcal{M}_\chi$ defined as

$$\mathcal{M}_\chi(u) := \mathcal{M}(u) + nJ_\chi(u). \tag{4.5}$$

Mabuchi [**Mab86, Mab87**] has proved that $\mathcal{M}$ is convex along smooth geodesics in $\mathcal{H}$. Smooth geodesic rays can be constructed using the flow of real holomorphic vector fields [**Mab87**, Theorem 3.5]. In general, even if u_0 and u_1 are in $\mathcal{H}$, the weak geodesic segment $(u_t)_{t\in[0,1]}$ is not necessarily in $\mathcal{H}$. It is thus desirable to have convexity of $\mathcal{M}$ along less regular geodesic segments. This is the content of a breakthrough of Berman-Berndtsson [**BB17**] (see also [**CLP16**] for a different proof): $t \mapsto \mathcal{M}(u_t)$ is continuous and convex in $[0,1]$ if u_0 and u_1 are in $\mathcal{H}$. An immediate consequence of this work is that a metric ω_u is cscK if and only if u minimizes $\mathcal{M}$. The problem of finding a cscK metric in $\mathcal{H}$ is thus reduced to finding a minimizer of $\mathcal{M}$ over $\mathcal{H}$. As explained above, we need to work in a space larger than $\mathcal{H}$ in order to use the compactness of sublevel sets of the K-energy $\mathcal{M}$. Building on the explicit formula of Chen and Tian, it was shown in [**BDL17**] that we can define $\mathcal{M}(u)$ for all $u \in \mathcal{E}^1$ and the extended functional $\mathcal{M}$ is convex along psh geodesics in $\mathcal{E}^1$.

THEOREM 4.8. *Let $p \geq 1$. Given a closed semipositive $(1,1)$-form χ, the extended functional $\mathcal{M}_\chi : (\mathcal{E}^p, d_p) \to (-\infty, +\infty]$, defined by (4.5), is the greatest d_p-lsc extension of $\mathcal{M}_\chi|_{\mathcal{H}}$. It is convex along psh geodesics in $\mathcal{E}^p$. Moreover, if $\chi > 0$, then $\mathcal{M}_\chi$ is strictly convex along psh geodesics in $\mathcal{E}^p$.*

IDEA OF THE PROOF FOR $\chi = 0$. We extend $\mathcal{M}$ over $\mathcal{E}^1$ by using the expression (4.5). Since the (twisted) Monge-Ampère energy is d_1-continuous and the relative entropy is lower semicontinuous with respect to weak convergence of measures, $\mathcal{M}$ is lower semi continuous on $\mathcal{E}^1$ (it may take the value $+\infty$). Any psh geodesic $(u_t)_{t\in[0,1]}$ in $\mathcal{E}^1$ can be approximated from above by $C^{1,\bar{1}}$-geodesics $(u_{t,j})$. The convexity of $\mathcal{M}$ along the approximating geodesics $(u_{t,j})$ was proved in [**BB17**]. The (twisted) Monge-Ampère energy part of $\mathcal{M}$ is continuous under decreasing sequences, while the entropy part is lower semicontinuous. Putting these together we get the convexity of $\mathcal{M}$ along (u_t).

To prove that this extension is the greatest d_p-lsc one, we need to show that, given $u \in \mathcal{E}^1$ with $\mathcal{M}(u) < +\infty$, there exists a sequence (u_j) of smooth Kähler potentials such that $d_p(u_j, u) \to 0$ and $\mathcal{M}(u_j) \to \mathcal{M}(u)$. When $p = 1$, the convergence of the entropy part is sufficient thanks to [**BBE$^+$19**]. When $p > 1$, the argument is more involved, and there are two different proofs.

The first one was given in [**BDL17**], where we first approximated u by solving Monge-Ampère equations

$$(\omega + dd^c u_\varepsilon)^n = e^{\varepsilon u_\varepsilon} f dV,$$

where $f = \omega_u^n/\omega^n$ is the density of ω_u^n. Then, for $\varepsilon > 0$ fixed we approximated f by smooth f_j and solved

$$(\omega + dd^c u_{\varepsilon,j})^n = e^{\varepsilon u_{\varepsilon,j}} f_j dV.$$

The second proof was given in [**DL20**], where we used the smoothing property of complex Monge-Ampère flows [**GZ17**], [**DNL17b**], to smooth out u while keeping track on its Monge-Ampère measure. □

LEMMA 4.9. *Assume $\varphi_0 \in \mathcal{E}^p$, $p \geq 1$, and $\mathcal{M}(\varphi_0) < +\infty$. Let $(\varphi_t)_{t>0}$ be the unique solution of the complex Monge-Ampère flow* (3.1) *with $F = 0$, $g = 1$, $\chi = 0$, starting from φ_0. Then $\mathcal{M}(\varphi_t) \to \mathcal{M}(\varphi_0)$ and $d_p(\varphi_t, \varphi_0)$ as $t \to 0$.*

A crucial ingredient used in the proof of Theorem 4.8 is the following compactness property obtained in [**BBE$^+$19**]: for any $C > 0$ the set

$$\{v \in \mathcal{E}^1 \ : \ d_1(v,0) \leq C \text{ and } \mathcal{M}(v) \leq C\}$$

is compact in $(\mathcal{E}^1, d_1)$. We have proved in [**DNGL21**] that this set is compact in $(\mathcal{E}^p, d_p)$ for any $p < \frac{n}{n-1}$:

THEOREM 4.10. *Let $\mu = (\omega + dd^c\varphi)^n = f\omega^n$ be a probability measure with finite entropy $\mathrm{Ent}_{\omega^n}(\mu) = \int_X f \log f \omega^n < +\infty$. Then*

$$\varphi \in \mathcal{E}^{\frac{n}{n-1}}(X, \omega).$$

Moreover, for any $C > 0$ and $p < \frac{n}{n-1}$, the set

$$\{v \in \mathcal{E}^p \ : \ d_p(v,0) \leq C \ and \ \mathcal{M}(v) \leq C\}$$

is compact in $(\mathcal{E}^p, d_p)$.

The proof of Theorem 4.10 relies on a Moser-Trudinger inequality which provides a strong integrability property of finite energy potentials.

THEOREM 4.11. *Fix $p > 0$. There exist positive constants $c, C > 0$ depending on X, ω, n, p such that, for all $\varphi \in \mathcal{E}^p(X, \omega)$ with $\sup_X \varphi = -1$,*

$$\int_X \exp\left(c|E_p(\varphi)|^{-1/n}|\varphi|^{1+\frac{p}{n}}\right)\omega^n \leq C.$$

Here $E_p(u) := \int_X |u|^p(\omega + dd^c u)^n$ is the pluricomplex energy previously studied in [**GZ07**] in the global context and in [**Ceg98**] in the local one.

Theorem 4.11 is an interesting variant of Trudinger's inequality on compact Kähler manifolds. The proof of Theorem 4.11 uses fine properties of quasi-psh envelopes, further exploited in [**GL21a, GL21b, GL21c**]. The case $p = 1$ settles a conjecture of Aubin (called Hypothèse fondamentale [**Aub84**]) which is motivated by the search for Kähler-Einstein metrics on Fano manifolds. The conjecture was previously proved by Berman-Berndtsson [**BB21**] under the assumption that the cohomology class of ω is integral.

The constant c in the above theorem is intimately related to the α-invariant of Tian. Our proof however does not provide an optimal one.

QUESTION 14. *What is the optimal constant c?*

6. Regularity of minimizers and coercivity of $\mathcal{M}$

It is important to note that there are obstructions to the existence of cscK metrics in a given cohomology class $\{\omega\}$. If ω is a cscK metric then, by Futaki and Calabi the Futaki invariant $F(\{\omega\})$ vanishes, and by Matsushima and Lichnerowitz the Lie algebra of G, the identity component of the automorphism group of X, is reductive.

Tian [**Tia94**], [**Tia00**] has conjectured that existence of cscK metrics in $\mathcal{H}$ should be equivalent to J_ω-properness of the K-energy $\mathcal{M}$. When G is trivial and X is Fano, the conjecture was verified in [**Tia97**], [**TZ00**] and strengthened by Phong-Song-Sturm-Weinkove [**PSSW08**] who established a stronger form saying that if a Kähler-Einstein metric exists then the K-energy grows at least linearly with respect to the J_ω-functional. When G is non-trivial, a modification of Tian's conjecture was proposed by Darvas-Rubinstein in [**DR17**]:

CONJECTURE 4.12. *There exists a cscK metric cohomologous to ω if and only if the K-energy is J_ω-coercive, i.e. there exists $\varepsilon > 0$ and $C > 0$ such that*

$$\mathcal{M}(u) \geq \varepsilon \inf_{g\in G} d_1(g.u, 0) - C,\ u \in \mathcal{H}_0,$$

where $\mathcal{H}_0 = \mathcal{H} \cap E^{-1}(0)$ is the space of normalized Kähler potentials.

Given $u \in \mathcal{H}_0$ and $g \in G$ the action $g.u$ is defined as follows. Since $g.\omega := g^*\omega$ is cohomologous to ω, by the dd^c-lemma there is a unique Kähler potential $g.u$ such that $g.\omega = \omega + dd^c(g.u)$ and $E(g.u) = 0$. In the conjecture above one can replace $d_1(g.u, 0)$ by $J_\omega(g.u)$ as they grow at the same rate.

In the same paper [**DR17**], Darvas-Rubinstein reduced the above conjecture to a conjecture on regularity of $\mathcal{E}^1$-minimizers of $\mathcal{M}$. In [**BDL20**] we solved the latter conjecture for cscK manifolds.

THEOREM 4.13. *Assume ω is a cscK metric. If $v \in \mathcal{E}^1$ minimizes the K-energy $\mathcal{M}$ then v is a smooth cscK potential. In particular, there exists $g \in G$ such that $g^*\omega_v = \omega$.*

As a consequence we obtain the following direction of Conjecture 4.12:

THEOREM 4.14. *If there exists a cscK metric cohomologous to ω then the K-energy is J_ω-coercive.*

IDEA OF THE PROOF OF THEOREM 4.13. Let (v_j) be a sequence of smooth Kähler potentials converging to v in d_1. We fix $\lambda > 0$ and introduce the twisted K-energy:

$$\mathcal{M}_\lambda := \mathcal{M} + nJ_{\lambda\omega_{v_j}}.$$

Since $\mathcal{M}_\lambda$ is strictly convex, there is a unique minimizer $v_j^\lambda \in \mathcal{E}^1 \cap E^{-1}(0)$. A direct computation then shows that v_j^λ minimizes $J_{\omega_{v_j}}$ over all $\mathcal{E}^1$-minimizers of $\mathcal{M}$. We next use the compactness result of [**BBE$^+$19**] and let $\lambda \to 0$ to obtain $v_j^0 \in \mathcal{E}^1$ minimizing $J_{\omega_{v_j}}$ over all $\mathcal{E}^1$-minimizers of $\mathcal{M}$. We also have the following crucial estimate

$$\mathcal{I}(v_j^0, v) \leq C\mathcal{I}(v_j, v).$$

The existence of a cscK metric on X implies that the group G is reductive. By the uniqueness argument in [**BB17**] we have that $v_j^0 = g_j.0$ for some $g_j \in G$. From the reductiveness of G and the estimate above, we can extract a subsequence of g_j converging to $g \in G$, giving $g.0 = v$. $\square$

Given the interest towards singular normal varieties, it seems very interesting to investigate the singular analogue of the properness conjecture.

QUESTION 15. *Let X be a normal complex Kähler space which admits a cscK metric ω. Is the K-energy $\mathcal{M}$ proper?*

7. K-polystability

The notion of K-stability goes back to work of Tian [**Tia97**], with generalizations and precisions made along the way by S. Donaldson [**Don02**], Li-Xu [**LX14**], Székélyhidi [**Szé14**] and many others.

Let $L \to X$ be an ample line bundle over a compact Kähler manifold (X, ω) such that $c_1(L) = \{\omega\}$. Then X is projective algebraic. A test configuration $(\mathcal{L}, \mathcal{X}, \pi, \rho)$ for (X, L) consists of a scheme $\mathcal{X}$ with a $\mathbb{C}^*$-equivariant flat surjective scheme morphism $\pi : \mathcal{X} \to \mathbb{C}$ and a relatively ample line bundle $\mathcal{L} \to \mathcal{X}$ with a $\mathbb{C}^*$-action $\tau \to \rho_\tau$ on $\mathcal{L}$ such that $(X_1, \mathcal{L}|_{X_1}) = (X, kL)$ for some $k > 1$. Without loss of generality we can assume that $k = 1$, by treating $\mathcal{L}$ as a $\mathbb{Q}$-line bundle. Following the findings of [**LX14**], we will always assume that $\mathcal{X}$ is normal, which automatically makes the projection π flat.

By analyzing the action of ρ restricted to global sections of $\mathcal{L}^r,\ r \geq 1$ on X_0, we can associate to $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ the Donalson–Futaki invariant $DF(\mathcal{X}, \mathcal{L})$. For details we refer to [**Szé14, Tho06**]. According to Phong–Sturm [**PS07, PS10**] to $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ one can also associate a $C^{1,\bar{1}}$ geodesic ray along which the slope of the K-energy is intimately related to the Donaldson-Futaki invariant.

Definition 4.15. We say that (X, L) is K-polystable if for any test configuration $(\mathcal{X}, \mathcal{L}, \pi, \rho)$ we have $DF(\mathcal{X}, \mathcal{L}) \geq 0$, with $DF(\mathcal{X}, \mathcal{L}) = 0$ if and only if $\mathcal{X}$ is a product.

We have proved in [**BDL20**] the following direction in the Yau-Tian-Donaldson conjecture:

Theorem 4.16. *Suppose $L \to X$ is a positive line bundle. If there exists a csck metric in the class $c_1(L)$, then (X, L) is K-polystable.*

When X is Fano, the result has been obtained by Berman [**Ber16**]. The proof of Theorem 4.16 relies on the properness Theorem 4.14 and an important formula for the slope of the K-energy along the associated geodesic ray [**PT09, PRS08, Tia12, BHJ19, SD18**].

As of this writing the other direction is still open but there has been remarkable progress due to Berman-Boucksom-Jonsson [**BBJ15**] and C. Li [**Li21**].

8. Geodesic stability

The collection of geodesic rays $(u_t)_t \in \mathcal{R}^1$ with $u_t \in \mathcal{H}^{1,\bar{1}},\ t \geq 0$ is denoted by $\mathcal{R}^{1,\bar{1}}$, and will be referred to as the set of *geodesic rays with $C^{1,\bar{1}}$ potentials.*

We have proved in [**DL20**] that $\mathcal{R}^\infty$ is d_p^c-dense in $\mathcal{R}^p$ for any $p \in [1, \infty)$. Also, we showed that $\mathcal{R}^{1,\bar{1}}$ is dense among rays with finite radial K-energy. In both cases one can approximate with converging radial K-energy:

Theorem 4.17. *Let $u \in \mathcal{R}^p$ with $p \in [1, \infty)$. The following hold:*
(i) There exists a sequence $(u^j) \in \mathcal{R}^\infty$ such that $u_t^j \searrow u_t,\ t \geq 0$, $d_p^c(u^j, u) \to 0$ and $\mathcal{M}(u^j) \to \mathcal{M}(u)$.
(ii) If $\mathcal{M}(u) < \infty$, then there exists a sequence $u^j \in \mathcal{R}^{1,\bar{1}}$ such that $u_t^j \searrow u_t,\ t \geq 0$, $d_p^c(u^j, u) \to 0$ and $\mathcal{M}(u^j) \to \mathcal{M}(u)$.

Here, the *radial K-energy* is defined for any ray $u \in \mathcal{R}^p$, and is given by the expression

$$\mathcal{M}(u) := \lim_{t \to \infty} \frac{\mathcal{M}(u_t)}{t}.$$

Idea of proof. As a first step in obtaining Theorem 4.17(i), we show that one can approximate by bounded geodesic rays with possibly diverging radial K-energy. The argument goes as follows. We first consider the Legendre transform ψ_τ of u_t, which is a concave curve of model potentials (but it is not a test curve). We approximate this curve by test curves ψ_τ^ε and take the inverse Legendre transform to get a bounded geodesic ray u^ε_t, which is decreasing as $\varepsilon \to 0$. To prove that $u^\varepsilon_t \searrow u_t$ we invoke the domination principle Lemma 1.5.

To obtain (i) in case $\mathcal{K}(u)$ is finite, a much more delicate construction will be needed, building on the relative Kołodziej type estimate Theorem 1.11. The idea is as follows. Given a ray (u_t) in $\mathcal{R}^1$, we fix large $t > 0$ and define a regularizing sequence $v^{(t,j)}$ by following the Monge-Ampère flow, [**GZ17**], [**DNL17b**]. We then consider the geodesic segment $(v^{t,j}_s)_{s\in[0,t]}$ connecting 0 to $u_{t,j}$. From a C^2 estimate due to He [**He15**] we can show that along a $C^{1,\bar 1}$ geodesic segment ϕ_t the function

$$[0,1] \ni t \to \text{ess sup}_X(\log(n + \Delta_\omega \phi_t) - B\phi_t) \in \mathbb{R}$$

is convex. This allows to get a uniform Laplacian bound for $v^{t,j}_s$ independent of t. We finally obtain our approximating rays by letting $t \to +\infty$.

□

We next turn to applications to existence of constant scalar curvature Kähler metrics in terms of geodesic stability, going back to Donaldson's conjectures in [**Don99**].

To start, we say that (X, ω) is *geodesically* L^p (respectively $C^{1,\bar 1}$) *semistable* if for any $u \in \mathcal{R}^p$ (respectively $\mathcal{R}^{1,\bar 1}$) we have that $\mathcal{M}(u) \geq 0$. As an immediate consequence of Theorem 4.17 we obtain that L^1-semistability and $C^{1,\bar 1}$-semistability are equivalent.

Recall that G, the identity component of the group of holomorphic automorphisms of X, acts on $\mathcal{E}^1_0 := \mathcal{E}^1 \cap E^{-1}(0)$, and one can introduce the following pseudo-metric on the orbits $\mathcal{E}^1_0/G$:

$$d_{1,G}(Gu_0, Gu_1) := \inf_{g\in G} d_1(u_0, g.u_1).$$

We will consider the space of *normalized rays* $\mathcal{R}^p_0$ (respectively $\mathcal{R}^{1,\bar 1}_0$), $p \in [1,\infty]$, where we restrict to rays $(u_t)_t \in \mathcal{R}^p$ (respectively $\mathcal{R}^{1,\bar 1}$) with $E(u_t) = 0,\ t \geq 0$.

By showing that minimizers of the K-energy on $\mathcal{E}^1$ are actually smooth csck potentials [**CC21b**, Theorem 1.5], Chen–Cheng have verified the last remaining condition of the existence/properness principle of [**DR17**], applied to the case of csck metrics. Together with the necessity result [**BDL20**, Theorem 1.5], their theorem shows that existence of csck metrics is equivalent with properness of $\mathcal{M}$ in the following sense: there exists $\delta, \gamma > 0$ such that

$$\mathcal{M}(u) \geq \delta d_{1,G}(G0, Gu) - \gamma, \quad u \in \mathcal{E}^1_0.$$

Clearly, $d_{1,G}(Gv_0, Gv_1) \leq d_1(v_0, v_1),\ v_0, v_1 \in \mathcal{E}^1$, and we say that $(u_t)_t \in \mathcal{R}^1$ is *G-calibrated* if the curve $t \to Gu_t$ is a $d_{1,G}$-geodesic with the same speed as $(u_t)_t$, i.e.,

$$d_{1,G}(Gu_0, Gu_t) = d_1(u_0, u_t), \quad t \geq 0.$$

Geometrically, $(u_t)_t$ is G-calibrated if it cuts each G-orbit inside $\mathcal{E}^1$ "perpendicularly". If G is trivial, every ray is G-calibrated.

Using our Theorem 4.17, the breakthrough of Chen–Cheng [**CC21a, CC21b**] together with [**Dar19**, Theorem 4.7] we have obtained in [**DL20**] the following $C^{1,\bar 1}$ uniform analogue of Donaldson's geodesic stability conjecture [**Don99**, Conjecture 12]:

THEOREM 4.18 ($C^{1,\bar 1}$ uniform geodesic stability). *The following are equivalent:*

(i) There exists a csck metric in $\mathcal{H}$.

(ii) There exists $\delta > 0$ *such that* $\mathcal{M}(u) \geq \delta \limsup_t \frac{d_{1,G}(G0,Gu_t)}{t}$ *for all* $u = (u_t) \in \mathcal{R}^{1,\bar 1}_0$.

(iii) $\mathcal{M}$ *is G-invariant and there exists* $\delta > 0$ *s.t.* $\mathcal{M}(u) \geq \delta d_1(0, u_1)$ *for all G-calibrated geodesic rays* $u = (u_t) \in \mathcal{R}^{1,\bar 1}_0$.

In the absence of non-trivial holomorphic vector fields, it is widely expected that uniform K-stability will be equivalent with existence of csck metrics (see [**CC21b**, Question 1.12], [**Bou18**, Conjecture 4.9]). Informally, uniform K-stability simply says that Theorem 4.18 holds for $C^{1,\bar{1}}$ rays that are induced by the so called test configurations of (X, ω). Closing the gap between L^1 uniform geodesic stability and uniform K-stability is the last remaining step in the variational program designed to attack the uniform Yau-Tian-Donaldson conjecture (see [**Bou18**, p.2]), with our Theorem 4.18 representing an intermediate step.

Outline of the proof. The G-invariance of $\mathcal{M}$ follows from the classical fact that $\mathcal{M}$ is affine along geodesic lines generated by holomorphic vector fields (see [**CC21b**, Lemma 3.3]). The implication $(i) \implies (ii)$ is a consequence of the resolution of Conjecture 4.12. By our Theorem 4.17, in (ii) we can replace $\mathcal{R}_0^{1,\bar{1}}$ by $\mathcal{R}_0^1$. To do the same thing in (iii), extra work is needed because our approximation scheme in Theorem 4.17 does not produce G-calibrated rays. The implications $(ii) \implies (i)$ and $(iii) \implies (i)$ then follow essentially from the breakthrough of Chen-Cheng [**CC21b**]. □

Given that the optimal regularity of geodesics is $C^{1,1}$ it is natural to investigate the $C^{1,1}$ version of Donaldson's geodesic stability conjecture:

Problem 6. *Prove the $C^{1,1}$ version of Theorem 4.18.*

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