

Workshop Paraplui Angers 2024  
Angers, 17-19 April 2024

All lectures and coffee breaks will take place in the room I001, LAREMA.

## Wednesday April 17

- 9h30-10h30: Alexandra Otiman (Aarhus)  
*Locally conformally Kähler geometry 1*
- 11h-12h: Vincent Guedj (Toulouse)  
*Characterization of the Fujiki class 1*
- 14h-15h: Laurent Meersseman (Angers)  
*Twistor spaces 1*
- 15h30-16h30: Federico Giusti (Zhejiang)  
*Chern-Ricci flat balanced orbifolds and crepant resolutions*

## Thursday 18 April

- 9h30-10h30: Nicolina Istrati (Angers)  
*Locally conformally Kähler geometry 2*
- 11h-12h: Vincent Guedj (Toulouse)  
*Characterization of the Fujiki class 2*
- 14h-15h: Laurent Meersseman (Angers)  
*Twistor spaces 2.*
- 15h30-16h30: Chung-Ming Pan (Toulouse)  
*Singular cscK metrics on smoothable varieties*
- 16h45-17h45: Daniele Angella (Florence)  
*Classification results for Locally Conformally Kähler threefolds*

## Friday 19 April

- 9h30-10h30: Ovidiu Preda (Bucharest)  
*Properties of Locally conformally Kähler spaces.*
- 11h-12h: Paul Gauduchon (Polytechnique)  
*Gravitational Instantons and Conformally Kaehler Geometry*

# Abstracts

## Characterization of the Fujiki class

We study the volumes of transcendental Bott-Chern  $(1, 1)$ -classes on an arbitrary compact complex manifold. We show that the latter belongs to the class  $\mathcal{C}$  of Fujiki if and only if its Monge-Ampère volumes have a uniform upper-bound and there exists a Bott-Chern class with positive volume. This yields a positive answer to a conjecture of Demailly-Paun-Boucksom. Joint work with S. Boucksom and C.H. Lu. References:

- S. Boucksom, V. Guedj, C.H. Lu, *Characterization of the Fujiki class*, Preprint 2024.
- V. Guedj and C.H. Lu, *Quasi-plurisubharmonic envelopes 2: Bounds on Monge-Ampère volumes*, *Algebr. Geom.* **9** (2022), no. 6, 688–713.

## Chern-Ricci flat balanced orbifolds and crepant resolutions

Given a non-Kähler Calabi-Yau orbifold with a finite family of isolated singularities endowed with a Chern-Ricci flat balanced metric, we show, via a gluing construction, that all its crepant resolutions admit Chern-Ricci flat balanced metrics, and discuss applications to the search of solutions for the Hull-Strominger system. We also describe the scenario of singular threefolds with ordinary double points, and see that similarly is possible to obtain balanced approximately Chern-Ricci flat metrics. References:

- F. Giusti, C. Spotti, *A Kummer construction for Chern-Ricci flat balanced manifolds*, arXiv:2301.11636.
- C. Arezzo, F. Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, *Acta Mathematica* **196.2** (2006), 179–228.
- O. Biquard, V. Minerbe, *A Kummer construction for gravitational instantons*, *Communications in mathematical physics* **308.3** (2011) 773–794.

## Locally conformally Kähler geometry

In the first part, we will present the different points of view on locally conformally Kähler (LCK) metrics, as well as the basic objects associated to such metrics. We will distinguish between different types of LCK metrics, defined in terms of the existence of a certain potential, and discuss some consequences of their existence. We end the first part by describing the three main classes of examples of LCK manifolds: the Hopf manifolds, the Oeljeklaus-Toma manifolds and the Kato manifolds.

In the second part, we focus on a specific class of LCK manifolds, the Vaisman manifolds. We will describe a general procedure of constructing such manifolds as torus bundles over projective manifolds, and give a characterization result of these. We will also introduce the main tool of the study of these manifolds, the canonical transversal Kähler foliation, present its different properties and its precise relation to the Vaisman metrics. As time permits, we will discuss the cohomology of these manifolds and different conjectures concerning it, and/or the automorphism group of a general LCK manifold and its relation to the Vaisman condition.

In the third part, we will talk about LCK analytic spaces. The first to introduce the definition of Kähler forms on complex analytic spaces was Grauert. This study

was continued later by Moishezon and Varouchas. We present how using Grauert’s idea, we can extend the definition of l.c.K. forms to complex analytic spaces, and which fundamental results of l.c.K. geometry of complex manifolds remain true in this more general setting. First, we show that the characterization theorem of l.c.K. manifolds via the universal cover remains true for l.c.K. spaces, without any additional assumption, and then we will see how this can be used for proving the existence of l.c.K. forms on spaces which can be mapped with discrete fibers onto an l.c.K. space. Then, we give a version of Vaisman’s theorem, which states that for locally irreducible compact spaces, pure l.c.K forms and Kähler forms cannot coexist, and also explain why the additional assumption of local irreducibility is necessary. Next, we show in which conditions the blow-up of a compact l.c.K. space admits l.c.K. forms, extending a known result by Ornea, Verbitsky and Vuletescu. Also, by relaxing the conditions in the definition of l.c.K. forms and naming them quasi-l.c.K. forms, we have a new class, of quasi-l.c.K. spaces, which is stable under modifications.

References:

- N. Istrati, *Existence criteria for special locally conformally Kähler metrics*, Ann. Mat. Pura Appl. (2019).
- K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (2005).
- L. Ornea, M. Verbitsky, *Principles of Locally Conformally Kähler Geometry*, arXiv:2208.07188.
- O. Preda, M. Stanciu, *Coverings of locally conformally Kähler complex spaces*, Math. Z. 298 (2021), no. 1-2, 639–651.
- O. Preda, M. Stanciu, *Vaisman’s theorem for lcK spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XXIV (2023), 2311–2321.
- O. Preda, M. Stanciu, *Blow-ups and modifications of lcK spaces*, Bull. Lond. Math. Soc. 55 (2023), 2481-2492.
- O. Preda, M. Stanciu, *Locally conformally Kähler spaces and proper open morphisms*, (2023) arXiv:2311.14372.
- I. Vaisman, *Generalized Hopf Manifolds*, Geometriae Dedicata (1982).

## Classification results for LCK threefolds

In this talk, we describe the structure of complex threefolds with algebraic dimension 2 in the case when they admit a metric that is locally conformal to Kähler metrics. In particular, we show that, under mild assumptions, every such manifold is essentially an elliptic fibration over a compact projective surface with isomorphic fibers. The talk is a joint collaboration with Maurizio Parton and Victor Vuletescu.

References:

- D. Angella, M. Parton, V. Vuletescu, *On locally conformally Kähler threefolds with algebraic dimension two*, Internat. Math. Res. Not. 2023, no. 5, 3948–3969.
- F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. 317 (2000), no. 1, 1–40.
- M. Brunella, *Locally conformally Kähler metrics on Kato surfaces*, Nagoya Math. J. 202 (2011), 77–81.
- K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin- New York, 1975.

## Twistor spaces

The purpose of these two presentations is to provide some publicity for twistor spaces, which constitute a very rich and important class of 3-manifolds that are non-Kählerian except in a few specific cases. Although these examples date back to the 1980s-1990s and had a significant impact at that time, they seem somewhat "forgotten" in recent times, particularly among non-Kählerian geometers. I will elaborate on their construction and some of their essential properties.

References:

- M. F. Atiyah, N.J. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*. Proc. Roy. Soc. London Ser. A362(1978), no.1711, 425–461.
- P. de Bartolomeis, A. Nannicini, *Introduction to differential geometry of twistor spaces*, Sympos. Math., XXXVIII Cambridge University Press, 1998, 91–160.
- N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc. (3) 43 (1981), no. 1, 133–150.
- S. M. Salamon, *Self-duality and twistor geometry*. Conference on Differential Geometry and Topology, Rend. Sem. Fac. Sci. Univ. Cagliari 58 (1988), 51–72.

## Gravitational Instantons and Conformally Kaehler Geometry

We describe some interesting families of ALF gravitational instantons with conical singularities. In particular, we completely understand the 5-dimensional family of Chen-Teo metrics and prove that only 4-dimensional subfamilies can be smoothly compactified so that the metric has conical singularities.

References:

- O. Biquard, P. Gauduchon, C. LeBrun, *Gravitational Instantons, Weyl Curvature, and Conformally Kaehler Geometry*. arXiv:2310.14387.
- O. Biquard, P. Gauduchon, *About a Family of ALF Instantons with Conical Singularities*. arXiv:2306.11110.

## Singular cscK metrics on smoothable varieties

Searching for canonical metrics in Kähler classes has been a central theme in Kähler geometry for decades. This talk aims to explain a method for investigating canonical metrics in families of singular varieties, employing relative versions of pluripotential theory and variational approach in families. We shall start by reviewing fundamental concepts and important properties within the variational picture of constant scalar curvature Kähler (cscK) metrics. I will then introduce notions of weak and strong topologies of quasi-plurisubharmonic functions in families, and explain several properties of entropy and Mabuchi functional extended to the family framework. Finally, I will demonstrate how these properties contribute to obtaining the stability of coercivity of Mabuchi functional for the family parameter and the construction of cscK metrics on smoothable varieties. Joint work with T. D. Tô and A. Trusiani.

References:

- X.X. Chen and J. Cheng, *On the constant scalar curvature Kähler metrics (I)—A priori estimates*, J. Amer. Math. Soc. **34** (2021), no. 4, 909–936.
- X.X. Chen and J. Cheng, *On the constant scalar curvature Kähler metrics (II)—Existence results*, J. Amer. Math. Soc. **34** (2021), no. 4, 937–1009.
- C.-M. Pan, T.-D. Tô, A. Trusiani, *Singular cscK metrics on smoothable varieties*, arXiv:2312.13653.